INTRODUCTION TO THE THEORY OF MHD EQUILIBRIA

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Chapter 1

The MHD Equations

A plasma is an electrically conducting fluid or gas consisting totally or partially of charged particles. The self-consistent theoretical description of a plasma is usually difficult because any electro-magnetic fields influence the motion of the plasma (charged particles) and at the same time the moving charges act as sources of the electro-magnetic field. A self-consistent description therefore leads to non-linear equations.

There are different levels of theoretical description for a plasma:

- N-body problem: equation of motion for each particle plus microscopic Maxwell's equations

Ensemble averaging

- Statistical Mechanics: Liouville's equation for N particle distribution function plus Maxwell's equations with averaged sources

BBGKY hierarchy

- Kinetic description: Boltzmann equation for one particle distribution functions plus Maxwell's equation with sources derived from integrating one particle distribution functions over velocity space

velocity moments

- (Multi-)Fluid description: Hydrodynamic equations including mean-field Lorentz force for each particle species + Maxwell's equations

quasi-neutrality, $v \ll c$, $m_i \gg m_e$, ...

- Single Fluid description (MHD)

For a detailed discussion of some of the steps see e.g. Balescu (1988).

Since the MHD equations have been discussed in many textbooks (e.g. Alfvén, 1950; Cowling, 1976; Mestel, 1999; Parker, 1979; Priest, 1982; Roberts, 1967; Sturrock, 1994) we will only state them here and discuss the physical meaning of the individual equations briefly.

a) Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \text{(mass conservation)} \tag{1.1}$$

b) Momentum conservation equation (equation of motion)

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \mathbf{j} \times \mathbf{B} - \nabla p - \rho \nabla \psi \tag{1.2}$$

c) Energy equation (various different forms possible)

$$\rho^\gamma \frac{\partial}{\partial t} \left(\frac{p}{\rho^\gamma} \right) + \mathbf{v} \cdot \nabla \left(\frac{p}{\rho^\gamma} \right) = -(\gamma - 1)\mathcal{L} \tag{1.3}$$

where

$$\mathcal{L} = \underbrace{\nabla \cdot \mathbf{q}}_{\text{heat flux}} + \overbrace{L_r}^{\text{radiative losses}} - \underbrace{\frac{\mathbf{j}^2}{\sigma}}_{\text{Ohmic heating}} - \overbrace{H.}^{\text{everything else}} \tag{1.4}$$

d) Ampère's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{1.5}$$

(displacement current neglected).

e) Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.6}$$

f) no magnetic monopoles

$$\nabla \cdot \mathbf{B} = 0 \tag{1.7}$$

g) Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{R} \tag{1.8}$$

$\mathbf{R}$ is a general right hand side and could contain Hall terms, electron and ion inertia terms etc. In MHD applications one often uses the simple form $\mathbf{R} = \eta \mathbf{j}$. Ohm's law can be regarded as a left-over of the electron equation of motion.

Chapter 2

Magnetohydrostatics

Magnetohydrostatics (MHS) is the theory of the static ($\partial/\partial t = 0$, $\mathbf{v} = \mathbf{0}$) equilibria of the MHD equations.

Under these assumptions we can see immediately that

a) the continuity equation is automatically satisfied

b) the momentum conservation equation becomes a force balance equation

$$\mathbf{0} = \mathbf{j} \times \mathbf{B} - \nabla p - \rho \nabla \psi \tag{2.1}$$

c) the energy equation becomes

$$\mathcal{L} = 0 \tag{2.2}$$

and will not be used at the moment.

d) Ampère's law remains unchanged

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{2.3}$$

e) Faraday's law

$$\nabla \times \mathbf{E} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{E} = \nabla \varphi \tag{2.4}$$

f) no magnetic monopoles

$$\nabla \cdot \mathbf{B} = 0 \tag{2.5}$$

g) Ohm's law (assuming $\mathbf{R} = \mathbf{0}$)

$$\mathbf{E} = \mathbf{0} \tag{2.6}$$

We finally end up with a set of three equations:

$$\begin{aligned} \mathbf{j} \times \mathbf{B} - \nabla p - \rho \nabla \psi &= \mathbf{0} \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned} \tag{2.7}$$

This set of equations has to be completed by an equation of state and assumptions about the temperature or an energy equation.

The fundamental question is:

Why is it useful to study MHS solutions ?

There are many answers to this question. I only state two which to me seem to be quite important.

a) From a fundamental point of view we can regard the MHD equations as a set of equations describing extremely complicated *dynamical systems*. In the study of dynamical systems it is always useful to start with a study of the simplest solutions. These are usually the stationary states and their bifurcation properties, in the MHD case the static equilibria.

b) From the point of view of modelling, many physical processes in plasma systems occur *slowly*, i.e. on time-scales which are much longer than the typical time-scale of the system. Such processes can be described systematically in the following way (see also Schindler and Birn, 1986). Let L be the length scale of the system, T the slow time scale of evolution and $v_A = B_0/\sqrt{\mu_0 \rho_0}$ a typical Alfvén speed. We then define the Alfvén time by $T_A = L/v_A$. The main assumption now is that

$$\frac{T_A}{T} = \frac{v}{v_A} = \varepsilon \ll 1. \tag{2.8}$$

We now normalize lengths by L, velocities by v, the magnetic field by B_0, the density by ρ_0, the pressure by p_0 and the gravitational potential by ψ_0. Normalised quantities will be denoted by a $\tilde{}$. We obtain

$$\varepsilon^2 \tilde{\rho} \left(\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{t}} + \tilde{\mathbf{v}} \cdot \tilde{\nabla} \tilde{\mathbf{v}} \right) = \tilde{\mathbf{j}} \times \tilde{\mathbf{B}} - 2\beta_p \tilde{\nabla} \tilde{p} - 2\beta_g \tilde{\rho} \tilde{\nabla} \tilde{\psi}. \tag{2.9}$$

Here, β_p is the ratio between plasma pressure and magnetic pressure, the so-called plasma beta, whereas β_g is a similar ratio between the gravitational energy density and the magnetic pressure. Both numbers measure the relative importance of pressure gradient and gravitational force with respect to the $\mathbf{j} \times \mathbf{B}$-force.

Since ε is assume to be small, we obtain to lowest order

$$\mathbf{0} = \tilde{\mathbf{j}} \times \tilde{\mathbf{B}} - 2\beta_p \tilde{\nabla} \tilde{p} - 2\beta_g \tilde{\rho} \tilde{\nabla} \tilde{\psi}. \tag{2.10}$$

So to lowest order we have the MHS force balance equation as fundamental equation and the time $\tilde{t}$ appears merely as a parameter. The fundamental importance of this *quasi-static approximation* lies in the fact that *sequences of MHS equilibria* can be used to model the slow evolution of plasma systems. However, one cautionary remark has to be made at this point: these sequences have to satisfy the constraints imposed by the other equations, especially Ohm's law and the continuity equations. These constraints usually lead to very complicated integro-differential problems which are difficult to solve.

Chapter 3

Symmetric Systems

We now assume that the solutions we want to find have spatial symmetries. The three most important symmetries are

- translational symmetry
- rotational symmetry
- helical symmetry .

Only in these three cases it is possible to reduce the MHS equations to one single elliptic second order partial differential equation (Solov'ev, 1967; Edenstrasser, 1980a,b). We will now show how this reduction can be achieved.

3.1 Translational Invariance with no External Forces

The basic equations are

$$\mathbf{j} \times \mathbf{B} - \nabla p - \rho \nabla \psi = \mathbf{0} \tag{3.1}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{3.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.3}$$

The usual (and probably best) way to tackle the problem is to write $\mathbf{B}$ in a form such that Eq. (3.3) is automatically satisfied. Let us assume that the invariant direction is the y-direction. Then

$$\frac{\partial}{\partial y} = 0. \tag{3.4}$$

Writing $\mathbf{B}$ as

$$\mathbf{B} = \nabla \times \mathbf{C} \tag{3.5}$$

with a vector potential $\mathbf{C}$ is one possibility, but not the best one in this case.

We rather write

$$\begin{aligned} \mathbf{B} &= \nabla A \times \mathbf{e}_y + B_y \mathbf{e}_y \\ &= \nabla \times (A\mathbf{e}_y) + B_y \mathbf{e}_y \end{aligned} \tag{3.6}$$

Here A and B_y depend only on x and z. Note that A is the y-component of the vector potential $\mathbf{C}$. A is usually called the flux function, because it is directly related to the magnetic flux in the xz-plane.

We also have

$$\mathbf{B} \cdot \nabla A = \underbrace{(\nabla A \times \mathbf{e}_y) \cdot \nabla A}_{=0} + B_y \underbrace{\mathbf{e}_y \cdot \nabla A}_{=0 \text{ since } \frac{\partial A}{\partial y}=0} = 0. \tag{3.7}$$

It follows that A is constant along magnetic field lines.

Magnetic field lines are defined by

$$\frac{d\mathbf{r}}{d\sigma} = \frac{\mathbf{B}\left(\mathbf{r}(\sigma)\right)}{|\mathbf{B}|} \tag{3.8}$$

and we get

$$\frac{d\mathbf{r}}{d\sigma} \cdot \nabla A = \frac{1}{|\mathbf{B}|} \frac{dA}{d\sigma} = 0. \tag{3.9}$$

The last equality shows that contours of A are projections of magnetic field lines onto the xz-plane.

Next we multiply Eq. (3.1) by $\mathbf{B}$ and get

$$\mathbf{B} \cdot (\mathbf{j} \times \mathbf{B}) = \mathbf{B} \cdot \nabla p = (\nabla A \times \mathbf{e}_y) \cdot \nabla p + B_y \underbrace{\mathbf{e}_y \cdot \nabla p}_{=0 \text{ since } \frac{\partial p}{\partial y}=0} . \tag{3.10}$$

The left hand side of this equation vanishes and we obtain

$$\mathbf{B} \cdot \nabla p = 0. \tag{3.11}$$

Equation (3.11) implies that the pressure p is constant along field lines and actually

$$(\nabla A \times \mathbf{e}_y) \cdot \nabla p = 0. \tag{3.12}$$

Written in terms of partial derivatives this equation becomes

$$\frac{\partial A}{\partial z} \frac{\partial p}{\partial x} - \frac{\partial A}{\partial x} \frac{\partial p}{\partial z} = 0. \tag{3.13}$$

We conclude that

$$p(x, z) = F(A(x, z)) \tag{3.14}$$

where F is an arbitrary positive function of A.

The current density follows from Eq. (3.2)

$$\begin{aligned}\nabla \times \mathbf{B} &= \nabla \times (\nabla \times (A\mathbf{e}_y)) + \nabla B_y \times \mathbf{e}_y \\ &= -\Delta A \mathbf{e}_y + \nabla \underbrace{[\nabla \cdot (A\mathbf{e}_y)]}_{=0} + \nabla B_y \times \mathbf{e}_y \\ &= \mu_0 \mathbf{j} \end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\mathbf{j} \times \mathbf{B} &= \frac{1}{\mu_0}\Big\{ -\Delta A \left[\mathbf{e}_y \times (\nabla A \times \mathbf{e}_y)\right] + \\ &\qquad (\nabla B_y \times \mathbf{e}_y) \times (\nabla A \times \mathbf{e}_y + B_y \mathbf{e}_y)\Big\} \\ &= \frac{1}{\mu_0}\Big\{ -\Delta A \nabla A + \underbrace{[(\nabla B_y \times \mathbf{e}_y)\cdot \mathbf{e}_y]}_{=0} \nabla A - \\ &\qquad [(\nabla B_y \times \mathbf{e}_y) \cdot \nabla A]\, \mathbf{e}_y - B_y \nabla B_y \Big\}. \end{aligned} \tag{3.16}$$

Since $\nabla p = dp/dA \nabla A$, we get

$$-\Delta A \nabla A - [(\nabla A \times \nabla B_y) \cdot \mathbf{e}_y]\, \mathbf{e}_y - B_y \nabla B_y = \mu_0 \frac{dp}{dA} \nabla A. \tag{3.17}$$

The y-component of this equation gives

$$\frac{\partial A}{\partial z}\frac{\partial B_y}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial B_y}{\partial z} = 0. \tag{3.18}$$

In the same way as for the pressure we conclude that

$$B_y(x,z) = G(A(x,z)), \tag{3.19}$$

implying that B_y is constant along field line projections onto the xz-plane.

The ∇A-component then gives

$$-\Delta A = \mu_0 \frac{dp}{dA} + B_y \frac{dB_y}{dA}. \tag{3.20}$$

This is the fundamental equation to be solved. It is usually called the Grad-Shafranov(-Schlüter) equation named after some of the people who first derived it (earlier versions have been found by Chandrasekhar, Dungey and others).

The pressure p and the magnetic field component B_y are free functions of A, which have to be chosen or fixed by additional information.

The right hand side of Eq. (3.20) is basically the y-component of the current density:

$$\mu_0 j_y = \mu_0 \frac{dp}{dA} + B_y \frac{dB_y}{dA}. \tag{3.21}$$

3.2 Rotational Invariance with no External Forces

In this case, we represent the magnetic field in cylindrical coordinates ϖ, ϕ and z by

$$\begin{aligned}
\mathbf{B} &= \nabla A \times \nabla \phi + B_\phi \mathbf{e}_\phi \\
&= \frac{1}{\varpi} \nabla A \times \mathbf{e}_\phi + B_\phi \mathbf{e}_\phi \\
&= -\frac{1}{\varpi}\frac{\partial A}{\partial z}\mathbf{e}_\varpi + \frac{1}{\varpi}\frac{\partial A}{\partial \varpi}\mathbf{e}_z + B_\phi \mathbf{e}_\phi \qquad (3.22)
\end{aligned}$$

Here A is *not* the ϕ-component of the vector potential, but A/ϖ is ! A does not depend on ϕ. A is again called the flux function and lines of constant A are magnetic field lines. Again we have that

$$\mathbf{B} \cdot \nabla p = 0 \qquad (3.23)$$

and

$$p(\varpi, z) = F(A(\varpi, z)). \qquad (3.24)$$

The current density is given by

$$\begin{aligned}
\nabla \times \mathbf{B} &= -\frac{\partial B_\phi}{\partial z}\mathbf{e}_\varpi - \left[\frac{1}{\varpi}\frac{\partial^2 A}{\partial z^2} + \frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial A}{\partial \varpi}\right)\right]\mathbf{e}_\phi + \frac{1}{\varpi}\frac{\partial}{\partial \varpi}\left(\varpi B_\phi\right)\mathbf{e}_z \\
&= \mu_0 \mathbf{j} \qquad (3.25)
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{j} \times \mathbf{B} &= \mathbf{j} \times \left(\frac{1}{\varpi}\nabla A \times \mathbf{e}_\phi + B_\phi \mathbf{e}_\phi\right) \\
&= \frac{1}{\varpi} j_\phi \nabla A - \frac{1}{\varpi}\left(\mathbf{j} \cdot \nabla A\right)\mathbf{e}_\phi + B_\phi \mathbf{j} \times \mathbf{e}_\phi \\
&= -\frac{1}{\mu_0 \varpi}\Bigg\{ \left[\frac{1}{\varpi}\frac{\partial^2 A}{\partial z^2} + \frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial A}{\partial \varpi}\right)\right]\nabla A - \\
&\qquad \left[\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\left(\varpi B_\phi\right)\frac{\partial A}{\partial z} - \frac{\partial B_\phi}{\partial z}\frac{\partial A}{\partial \varpi}\right]\mathbf{e}_\phi + \\
&\qquad B_\phi \nabla\left(\varpi B_\phi\right)\Bigg\} \\
&= \frac{dp}{dA}\nabla A. \qquad (3.26)
\end{aligned}$$

Looking at the ϕ-component of this equation we see that

$$\begin{aligned}
\frac{1}{\varpi}\left[\frac{\partial}{\partial \varpi}\left(\varpi B_\phi\right)\frac{\partial A}{\partial z} - \frac{\partial}{\partial z}\left(\varpi B_\phi\right)\frac{\partial A}{\partial \varpi}\right] &= \frac{1}{\varpi}\mathbf{B} \cdot \nabla\left(\varpi B_\phi\right) \\
&= 0. \qquad (3.27)
\end{aligned}$$

It follows that

$$b_\phi(\varpi, z) = \varpi B_\phi(\varpi, z) = G(A(\varpi, z)). \tag{3.28}$$

This allows us to write the rest of the equations as

$$-\frac{1}{\mu_0 \varpi}\left[\frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial A}{\partial \varpi}\right) + \frac{1}{\varpi}\frac{\partial^2 A}{\partial z^2}\right]\nabla A - \frac{1}{\mu_0 \varpi^2} b_\phi \frac{db_\phi}{dA}\nabla A = \frac{dp}{dA}\nabla A. \tag{3.29}$$

Since all terms are in the direction of ∇A we finally obtain the Grad-Shafranov(-Schlüter) equation for rotational symmetry in cylindrical coordinates

$$-\nabla \cdot \left(\frac{1}{\varpi^2}\nabla A\right) = \mu_0 \frac{dp}{dA} + \frac{1}{\varpi^2} b_\phi \frac{db_\phi}{dA} \tag{3.30}$$

This equation should *not* be confused with the Grad-Shafranov(-Schlüter) equation in cylindrical coordinates but with translational symmetry, i.e. $\partial/\partial z = 0$ instead of $\partial/\partial\phi = 0$!

3.3 Helical Invariance with no External Forces

We have so far treated two special symmetries which seem to be quite "natural". The obvious question to ask is:

Are there other more general symmetries for which the MHS equations can be reduced to a single elliptic PDE?

The answer to that question has two parts

a) Formulate the MHD equations in a general coordinate system ξ^1, ξ^2, ξ^3 and use tensor calculus to carry out the manipulations, see e.g. Edenstrasser (1980a).

b) Assume that the physical quantities do not depend on ξ^3, say. This implies restrictions on the coordinate system, see e.g. Edenstrasser (1980b).

The most general invariance for which the reduction to a single elliptic PDE is possible is helical invariance. Here the coordinate lines along which the physical quantities have to be invariant have the form of helices

$$\mathcal{H}(\varpi, \phi, z) = \{\varpi, \phi, z | \varpi = \text{const.}, u = n\phi - kz = \text{const.}\} \tag{3.31}$$

with n and k real constants. So, in this case, ϖ and u are the two independent variables upon which all the other quantities depend. We start our analysis again by looking at the condition

$$\nabla \cdot \mathbf{B} = 0 \tag{3.32}$$

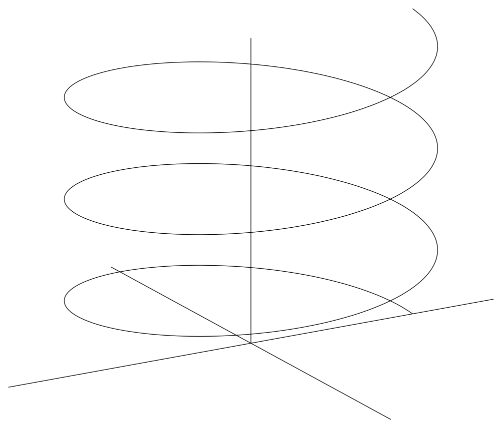

Figure 3.1: Helical line

which we rewrite in the following way in cylindrical coordinates ϖ, ϕ, z:

$$\begin{aligned}\frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi B_\varpi)+\frac{1}{\varpi}\frac{\partial B_\phi}{\partial\phi}+\frac{\partial B_z}{\partial z} &= \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi B_\varpi)+\frac{1}{\varpi}\frac{\partial u}{\partial\phi}\frac{\partial B_\phi}{\partial u}+\frac{\partial u}{\partial z}\frac{\partial B_z}{\partial u} \\ &= 0 \end{aligned} \quad (3.33)$$

because the components of $\mathbf{B}$ depend only on ϖ and u.

Substituting the definition of u into Eq. (3.33), we get

$$\begin{aligned}\frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi B_\varpi)+\frac{n}{\varpi}\frac{\partial B_\phi}{\partial u}-k\frac{\partial B_z}{\partial u} &= \frac{1}{\varpi}\left[\frac{\partial}{\partial\varpi}(\varpi B_\varpi)+\frac{\partial}{\partial u}(nB_\phi-k\varpi B_z)\right] \\ &= 0 \end{aligned} \quad (3.34)$$

because ϖ and u are orthogonal coordinates. This equation is solved by introducing a flux function $A(\varpi,u)$ in the following way:

$$\varpi B_\varpi = \frac{\partial A}{\partial u} \quad (=\varpi\mathbf{e}_\varpi\cdot\mathbf{B}) \quad (3.35)$$

$$nB_\phi-k\varpi B_z = -\frac{\partial A}{\partial\varpi} \quad (=\varpi\nabla u\cdot\mathbf{B}). \quad (3.36)$$

These two equations relate the three components of $\mathbf{B}$, expressed in cylindrical coordinates, with each other, and leaves one independent component of $\mathbf{B}$ as in the previous cases. This is the component in the direction $\mathbf{e}_\varpi\times\nabla u$.

By taking the scalar product of the momentum balance equation with $\mathbf{B}$ we again get

$$\mathbf{B}\cdot\nabla p = B_\varpi\frac{\partial p}{\partial\varpi}+B_\phi\frac{\partial u}{\partial\phi}\frac{\partial p}{\partial u}+B_z\frac{\partial u}{\partial z}\frac{\partial p}{\partial u}$$

$$
\begin{aligned}
&= B_\varpi \frac{\partial p}{\partial \varpi} + \frac{1}{\varpi}\left(nB_\phi - k\varpi B_z\right)\frac{\partial p}{\partial u} \\
&= 0
\end{aligned} \tag{3.37}
$$

Substituting our two equations for $\mathbf{B}$ in terms of A, we get

$$
\frac{1}{\varpi}\left(\frac{\partial A}{\partial u}\frac{\partial p}{\partial r} - \frac{\partial A}{\partial r}\frac{\partial p}{\partial u}\right) = 0. \tag{3.38}
$$

As in the previous cases we may conclude that

$$
p(r,u) = F(A(r,u)) \tag{3.39}
$$

and

$$
\nabla p = \frac{dF}{dA}\nabla A \tag{3.40}
$$

Next we consider the individual components of the momentum balance equation. First we calculate the current density $\mathbf{j}$

$$
\begin{aligned}
\mu_0 \mathbf{j} &= \nabla \times \mathbf{B} \\
&= \left(\frac{1}{\varpi}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right)\mathbf{e}_\varpi + \left(\frac{\partial B_\varpi}{\partial z} - \frac{\partial B_z}{\partial \varpi}\right)\mathbf{e}_\phi + \\
&\qquad \left[\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\left(\varpi B_\phi\right) - \frac{1}{\varpi}\frac{\partial B_\varpi}{\partial \phi}\right]\mathbf{e}_z \\
&= \left(\frac{n}{\varpi}\frac{\partial B_z}{\partial u} + k\frac{\partial B_\phi}{\partial u}\right)\mathbf{e}_\varpi + \left(-k\frac{\partial B_\varpi}{\partial u} - \frac{\partial B_z}{\partial \varpi}\right)\mathbf{e}_\phi + \\
&\qquad \left[\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\left(\varpi B_\phi\right) - \frac{n}{\varpi}\frac{\partial B_\varpi}{\partial u}\right]\mathbf{e}_z
\end{aligned} \tag{3.41}
$$

and the $\mathbf{j} \times \mathbf{B}$ force follows as

$$
\begin{aligned}
\mathbf{j} \times \mathbf{B} &= \frac{1}{\mu_0}\Bigg\{\Bigg[\left(-k\frac{\partial B_\varpi}{\partial u} - \frac{\partial B_z}{\partial \varpi}\right)B_z \\
&\qquad - \left[\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\left(\varpi B_\phi\right) - \frac{1}{\varpi}\frac{\partial B_\varpi}{\partial \phi}\right]B_\phi\Bigg]\mathbf{e}_\varpi + \\
&\quad \Bigg[\left[\frac{1}{\varpi}\frac{\partial}{\partial \varpi}\left(\varpi B_\phi\right) - \frac{n}{\varpi}\frac{\partial B_\varpi}{\partial u}\right]B_\varpi \\
&\qquad - \left(\frac{n}{\varpi}\frac{\partial B_z}{\partial u} + k\frac{\partial B_\phi}{\partial u}\right)B_z\Bigg]\mathbf{e}_\phi + \\
&\quad \Bigg[\left(\frac{n}{\varpi}\frac{\partial B_z}{\partial u} + k\frac{\partial B_\phi}{\partial u}\right)B_\phi \\
&\qquad - \left(-k\frac{\partial B_\varpi}{\partial u} - \frac{\partial B_z}{\partial \varpi}\right)B_\varpi\Bigg]\mathbf{e}_z\Bigg\}
\end{aligned} \tag{3.42}
$$

We first look at the ϕ and the z-component of the force balance equation. The ϕ-component gives

$$\frac{1}{\varpi}\left[\frac{\partial}{\partial\varpi}(\varpi B_\phi) - n\frac{\partial B_\varpi}{\partial u}\right]B_\varpi - \frac{1}{\varpi}\left(n\frac{\partial B_z}{\partial u} + k\varpi\frac{\partial B_\phi}{\partial u}\right)B_z = \mu_0\frac{n}{\varpi}\frac{dp}{dA}\frac{\partial A}{\partial u} \tag{3.43}$$

and the z-component is

$$\frac{1}{\varpi}\left(n\frac{\partial B_z}{\partial u} + k\varpi\frac{\partial B_\phi}{\partial u}\right)B_\phi + \left(k\frac{\partial B_\varpi}{\partial u} + \frac{\partial B_z}{\partial\varpi}\right)B_\varpi = -\mu_0 k\frac{dp}{dA}\frac{\partial A}{\partial u} \tag{3.44}$$

We multiply the ϕ-equation with $k\varpi$ and the z-equation with n, add the two and get

$$\begin{aligned} k\left[\frac{\partial}{\partial\varpi}(\varpi B_\varpi) - n\frac{\partial B_\phi}{\partial u}\right]B_\varpi - k\left[n\frac{\partial B_z}{\partial u} + k\varpi\frac{\partial B_\phi}{\partial u}\right]B_z \quad + \\ \frac{n}{\varpi}\left[n\frac{\partial B_z}{\partial u} + k\varpi\frac{\partial B_\phi}{\partial u}\right]B_\phi + n\left[k\frac{\partial B_\varpi}{\partial u} + \frac{\partial B_z}{\partial\varpi}\right]B_\varpi \quad = \quad 0. \end{aligned} \tag{3.45}$$

It follows that

$$\begin{aligned} B_\varpi\left[\frac{\partial}{\partial\varpi}\left(\underbrace{k\varpi B_\phi + nB_z}_{=B_v}\right) - kn\frac{\partial B_\varpi}{\partial u} + nk\frac{\partial B_\varpi}{\partial u}\right] \quad + \\ \left(\frac{n}{\varpi}B_\phi - kB_z\right)\frac{\partial}{\partial u}(k\varpi B_\phi + nB_z) \quad = \quad 0. \end{aligned} \tag{3.46}$$

and therefore

$$\varpi B_\varpi\frac{\partial B_v}{\partial\varpi} + (nB_\phi - k\varpi B_z)\frac{\partial B_v}{\partial u} = 0. \tag{3.47}$$

Substituting

$$\varpi B_\varpi = \frac{\partial A}{\partial u} \tag{3.48}$$

and

$$nB_\phi - k\varpi B_z = -\frac{\partial A}{\partial\varpi} \tag{3.49}$$

we get

$$\frac{\partial A}{\partial u}\frac{\partial B_v}{\partial\varpi} - \frac{\partial A}{\partial\varpi}\frac{\partial B_v}{\partial u} = 0. \tag{3.50}$$

From this equation we obtain finally that

$$B_v(\varpi, u) = h(A(\varpi, u)). \tag{3.51}$$

From the two equations

$$nB_\phi - k\varpi B_z = -\frac{\partial A}{\partial\varpi} \tag{3.52}$$

$$k\varpi B_\phi + nB_z = h(A) \tag{3.53}$$

we derive the following expressions for B_ϕ and B_z:

$$B_\phi = \frac{1}{n^2+k^2\varpi^2}\left(k\varpi h(A) - n\frac{\partial A}{\partial\varpi}\right) \quad (3.54)$$

$$B_z = \frac{1}{n^2+k^2\varpi^2}\left(k\varpi\frac{\partial A}{\partial\varpi} + nh(A)\right) \quad (3.55)$$

To derive the Grad-Shafranov equation for helically symmetric equilibria, we now have a look at the ϖ-component of the momentum balance:

$$\left(-k\frac{\partial B_\varpi}{\partial u} - \frac{\partial B_z}{\partial\varpi}\right)B_z - \left(\frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi B_\phi) - \frac{n}{\varpi}\frac{\partial B_\varpi}{\partial u}\right)B_\phi = \mu_0\frac{dp}{dA}\frac{\partial A}{\partial\varpi} \quad (3.56)$$

We substitute our expression for B_ϖ into the $\partial/\partial u$ derivatives and get

$$-\frac{1}{\varpi^2}\frac{\partial^2 A}{\partial u^2}\left(\underbrace{k\varpi B_z - nB_\phi}_{\frac{\partial A}{\partial\varpi}}\right) - B_z\frac{\partial B_z}{\partial\varpi} - \frac{B_\phi}{\varpi}\frac{\partial}{\partial\varpi}(\varpi B_\phi) = \mu_0\frac{dp}{dA}\frac{\partial A}{\partial\varpi}. \quad (3.57)$$

We rewrite this equation as

$$-\frac{1}{\varpi^2}\frac{\partial^2 A}{\partial u^2}\frac{\partial A}{\partial\varpi} - \left[\frac{1}{2}\frac{\partial}{\partial\varpi}\left(B_z^2 B_\phi^2\right) + \frac{B_\phi^2}{\varpi}\right] = \mu_0\frac{dp}{dA}\frac{\partial A}{\partial\varpi}. \quad (3.58)$$

The strategy now is to substitute the expression for B_ϕ and B_z into this equation and to manipulate it into a form that the term inside the brackets also has a factor $\partial A/\partial\varpi$. To do this we first calculate $B_z^2 + B_\phi^2$.

$$\begin{aligned} B_z^2 + B_\phi^2 &= \frac{1}{(n^2+k^2\varpi^2)^2}\left[k^2\varpi^2 h^2 - 2kn\varpi h\frac{\partial A}{\partial\varpi} + n^2\left(\frac{\partial A}{\partial\varpi}\right)^2\right. \\ &\qquad \left. + k^2\varpi^2\left(\frac{\partial A}{\partial\varpi}\right)^2 + 2kn\varpi h\frac{\partial A}{\partial\varpi} + n^2 h^2\right] \\ &= \frac{1}{n^2+k^2\varpi^2}\left[h^2 + \left(\frac{\partial A}{\partial\varpi}\right)^2\right] \quad (3.59) \end{aligned}$$

Then we obtain for the term in brackets

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial\varpi}\left(B_\phi^2 + B_z^2\right) + \frac{1}{\varpi}B_\phi^2 = \\ &\quad -\frac{k^2\varpi}{(n^2+k^2\varpi)^2}\left[h^2 + \left(\frac{\partial A}{\partial\varpi}\right)^2\right] + \frac{1}{n^2+k^2\varpi^2}\left(h\frac{dh}{dA}\frac{\partial A}{\partial\varpi} + \frac{\partial^2 A}{\partial\varpi^2}\frac{\partial A}{\partial\varpi}\right) \\ &\quad + \frac{1}{(n^2+k^2\varpi^2)^2}\left[k^2\varpi h^2 - 2knh\frac{\partial A}{\partial\varpi} + \frac{n^2}{\varpi}\left(\frac{\partial A}{\partial\varpi}\right)^2\right] \\ &= \frac{1}{n^2+k^2\varpi^2}\left(\frac{\partial^2 A}{\partial\varpi^2} - \frac{k^2\varpi - n^2}{n^2+k^2\varpi^2}\frac{\partial A}{\partial\varpi} + \right. \\ &\qquad \left. h\frac{dh}{dA} - \frac{2kn}{n^2+k^2+\varpi^2}h\right)\frac{\partial A}{\partial\varpi} \quad (3.60) \end{aligned}$$

We now substitute this expression back into the ϖ-component of the momentum balance equation and get, omitting the factor $\partial A/\partial\varpi$

$$-\frac{1}{\varpi^2}\frac{\partial^2 A}{\partial u^2}-\frac{1}{n^2+k^2\varpi^2}\frac{\partial^2 A}{\partial\varpi^2}+\frac{k^2\varpi^2-n^2}{\varpi(n^2+k^2\varpi^2)}\frac{\partial A}{\partial\varpi}= \mu_0\frac{dp}{dA}+\frac{1}{n^2+k^2+\varpi^2}h\frac{dh}{dA}-\frac{2kn}{(n^2+k^2\varpi^2)^2}h \quad (3.61)$$

We can write this in a more compact form if we multiply Eqn. (3.61) by $n^2+k^2\varpi^2$ and introduce the Laplace operator in the following form

$$\begin{aligned}\Delta A &= \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial A}{\partial\varpi}\right)+\frac{1}{\varpi^2}\frac{\partial^2 A}{\partial\phi^2}+\frac{\partial^2 A}{\partial z^2}\\ &= \frac{\partial^2 A}{\partial\varpi^2}+\frac{1}{\varpi}\frac{\partial A}{\partial\varpi}+\frac{1}{\varpi^2}\frac{\partial u}{\partial\phi}\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial\phi}\frac{\partial A}{\partial u}\right)+\frac{\partial u}{\partial z}\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial z}\frac{\partial A}{\partial u}\right)\\ &= \frac{\partial^2 A}{\partial\varpi^2}+\frac{1}{\varpi}\frac{\partial A}{\partial\varpi}+\frac{n^2}{\varpi^2}\frac{\partial^2 A}{\partial u^2}+k^2\frac{\partial^2 A}{\partial u^2}\\ &= \frac{\partial^2 A}{\partial\varpi^2}+\frac{1}{\varpi}\frac{\partial A}{\partial\varpi}+\frac{n^2+k^2\varpi^2}{\varpi^2}\frac{\partial^2 A}{\partial u^2}. \end{aligned} \quad (3.62)$$

This allows us to write the left hand side of Eq. (3.61) as

$$\begin{aligned}-\Delta A+\frac{1}{\varpi}\frac{\partial A}{\partial\varpi}+\frac{k^2\varpi^2-n^2}{\varpi(n^2+k^2\varpi^2)}\frac{\partial A}{\partial\varpi} &= -\Delta A+\frac{n^2+k^2\varpi^2+k^2\varpi^2-n^2}{\varpi(n^2+k^2\varpi^2)}\frac{\partial A}{\partial\varpi}\\ &= -\Delta A+\frac{2k^2\varpi}{n^2+k^2\varpi^2}\frac{\partial A}{\partial\varpi}\\ &= -\Delta A+\nabla\ln(n^2+k^2\varpi^2)\cdot\nabla A\\ &= \mathcal{L}A \end{aligned} \quad (3.63)$$

Then the Grad-Shafranov-equation looks like this :

$$\mathcal{L}A=(n^2+k^2\varpi^2)\mu_0\frac{dp}{dA}+h\frac{dh}{dA}-\frac{2kn}{n^2+k^2\varpi^2}h. \quad (3.64)$$

Another way to write Eq. (3.61) in a compact form is to not multiply the equation by $n^2+k^2\varpi^2$ and see that

$$\begin{aligned}&-\nabla\cdot\left(\frac{1}{n^2+k^2\varpi^2}\nabla A\right)\\ &= -\frac{1}{n^2+k^2\varpi^2}\Delta A+\frac{2k^2\varpi}{(n^2+k^2\varpi^2)^2}\frac{\partial A}{\partial\varpi}\\ &= -\frac{1}{n^2+k^2\varpi^2}\left(\frac{\partial^2 A}{\partial\varpi^2}\frac{1}{\varpi}\frac{\partial A}{\partial\varpi}+\frac{n^2+k^2\varpi^2}{\varpi^2}\frac{\partial^2 A}{\partial u^2}\right)+\frac{2k^2\varpi}{n^2+k^2\varpi^2}\frac{\partial A}{\partial\varpi}\\ &= -\frac{1}{\varpi^2}\frac{\partial^2 A}{\partial u^2}-\frac{1}{n^2+k^2\varpi^2}\frac{\partial^2 A}{\partial\varpi^2}+\frac{2k^2\varpi^2-n^2-k^2\varpi^2}{\varpi(n^2+k^2\varpi^2)}\frac{\partial A}{\partial\varpi}\\ &= -\frac{1}{\varpi^2}\frac{\partial^2 A}{\partial u^2}-\frac{1}{n^2+k^2\varpi^2}\frac{\partial^2 A}{\partial\varpi^2}+\frac{k^2\varpi^2-n^2}{\varpi(n^2+k^2\varpi^2)}\frac{\partial A}{\partial\varpi} \end{aligned} \quad (3.65)$$

which is equal to the left hand side of the Eq. (3.61).

In this form the GS-equation reads

$$-\nabla \cdot \left(\frac{1}{n^2 + k^2 \varpi^2} \nabla A \right) = \mu_0 \frac{dp}{dA} + \frac{1}{n^2 + k^2 \varpi^2} h \frac{dh}{dA} - \frac{2kn}{n^2 + k^2 \varpi^2} h \quad (3.66)$$

This form is particularly useful to see that the two cases of translational symmetry and rotational symmetry are only special cases of the more general helical symmetry. We recover the case of translational symmetry for $k = 0$ and $n = 1$ and the case of rotational symmetry for $k = 1$ and $n = 0$.

Notice that the case of translational symmetry here implies $\partial/\partial z = 0$ instead of $\partial/\partial y = 0$ as used before in Section 3.1. We recover this case after suitably renaming the coordinates.

3.4 The Inclusion of External Forces

So far we have not included external forces. We will now show, how this can be done in the case of translational invariance for an external gravitational field and in the case of rotational invariance for the case of a plasma which is rotating. Other cases can be treated similarly.

3.4.1 External gravity in Cartesian coordinates

We assume here that the external force has a potential Ψ and can be written as

$$\mathbf{f} = -\rho \nabla \Psi \quad (3.67)$$

with Ψ a known function of space. The force balance equation then is

$$\mathbf{j} \times \mathbf{B} - \nabla p - \rho \nabla \Psi = 0. \quad (3.68)$$

Again we write

$$\mathbf{B} = \nabla A \times \mathbf{e}_y + B_y \mathbf{e}_y. \quad (3.69)$$

But now the scalar product of Eq. (3.68) with $\mathbf{B}$ gives

$$\mathbf{B} \cdot \nabla p = -\rho \mathbf{B} \cdot \nabla \Psi \quad (3.70)$$

and the pressure will not be constant along field lines if $\nabla \Psi$ has a component acting along $\mathbf{B}$.

Physically, it is intuitively clear why this must be the case. Since the $\mathbf{j} \times \mathbf{B}$-force only acts *across* the field, any component of force *along* the field has to be balanced by a pressure gradient *along* the field. But how can we proceed now?

From the derivation of the GS-equation without external forces, we know, that we can write

$$\mathbf{j} \times \mathbf{B} = \frac{1}{\mu_0} \left\{ -\Delta A \nabla A - \left[(\nabla A \times \nabla B_y) \cdot \mathbf{e}_y \right] \mathbf{e}_y - B_y \nabla B_y \right\} \quad (3.71)$$

Since neither ∇p nor $\rho\nabla\Psi$ are allowed to have a y-component because of our symmetry assumption, we again get the result that

$$B_y(x,z) = G(A(x,z)) \tag{3.72}$$

and therefore

$$\mathbf{j}\times\mathbf{B} = \frac{1}{\mu_0}\left(-\Delta A - B_y\frac{dB_y}{dA}\right)\nabla A \tag{3.73}$$

The force balance equation now has the form

$$-\frac{1}{\mu_0}\left(\Delta A + B_y\frac{dB_y}{dA}\right)\nabla A - \nabla p - \rho\nabla\Psi \tag{3.74}$$

We now use the following argument. Each of the three vector fields ∇A, ∇p and $\nabla\Psi$ has only two components, namely in the x-z-plane. It follows that only two of these vector fields can be linearly independent.

We now assume that ∇A and $\nabla\Psi$ are linearly independent (apart from sets of measure zero). In the case that ∇A and $\nabla\Psi$ are linearly dependent, we have the highly special case that the external force is everywhere perpendicular to $\mathbf{B}$, which restricts $\mathbf{B}$ considerably. This special case can be treated in the same way as the case without external forces, but leads to a slightly different right-hand-side

$$-\Delta A = \mu_0\left(\frac{dp}{dA} + \rho\frac{d\Psi}{dA}\right) \tag{3.75}$$

We then need an equation for ρ to complete the problem, e.g. the ideal gas law with constant temperature T

$$p = R\rho T = c_s^2\rho \tag{3.76}$$

where c_s = constant is the sound velocity or the adiabatic relation between p and ρ

$$p = K\rho^\gamma. \tag{3.77}$$

Let us now return to the more general case with ∇A and $\nabla\Psi$ linearly independent. Then we can write

$$\nabla p = p_A\nabla p + p_\Psi\nabla\Psi \tag{3.78}$$

with functions p_A and p_ψ as coefficients. Since

$$\nabla\times\nabla p = \mathbf{0} \tag{3.79}$$

we get

$$\nabla p_A\times\nabla A + \nabla p_\Psi\times\nabla\Psi = \mathbf{0} \tag{3.80}$$

It is easy to verify that this equation is fulfilled if $p = F(A, \Psi)$, $p_A = \partial F/\partial A$ and $p_\Psi = \partial F/\partial \Psi$. Substituting these equations into Eq. (3.80), we get

$$\left(\frac{\partial^2 F}{\partial A^2}\nabla A + \frac{\partial^2 F}{\partial \psi \partial A}\nabla \Psi\right) \times \nabla A + \left(\frac{\partial^2 F}{\partial A \partial \Psi}\nabla A + \frac{\partial^2 F}{\partial A^2}\nabla \Psi\right) \times \nabla \Psi \quad = \frac{\partial^2 F}{\partial A \partial \Psi}\left(\nabla \Psi \times \nabla A + \nabla A \times \nabla \Psi\right) = \mathbf{0}. \tag{3.81}$$

With this general result for p, we now write the force balance equation as

$$\left[-\frac{1}{\mu_0}\left(\Delta A + B_y \frac{dB_y}{dA}\right) - \frac{\partial p}{\partial A}\right]\nabla A - \left(\frac{\partial p}{\partial \Psi} + \rho\right)\nabla \Psi = 0. \tag{3.82}$$

Since ∇A and $\nabla \Psi$ are linearly independent their coefficients must vanish and we get

$$-\Delta A = \mu_0 \frac{\partial p}{\partial A} + B_y \frac{dB_y}{dA} \tag{3.83}$$

$$\frac{\partial p}{\partial \Psi} = -\rho. \tag{3.84}$$

Obviously, to complete the problem we need to know something about ρ. There are several ways of providing this additional information.

a) Fix ρ as a function of A and Ψ and integrate Eq. (3.84) to get p as function of A and Ψ. The disadvantage of this approach is that it usually leads to unphysical equations of state.

b) Assume a specific equation of state and specify the temperature as a function of A and Ψ.

   Examples:

   (a) Ideal gas with $p = R\rho T$ with $T = T(A, \Psi)$ given. It follows that

   $$\frac{\partial p}{\partial \psi} = -\frac{p}{RT} \tag{3.85}$$

   and integrating once we get

   $$p = p_0(A)\exp\left(-\int_{\Psi_0}^{\Psi} \frac{d\Psi'}{RT(A, \Psi')}\right). \tag{3.86}$$

   In the special case of an isothermal ideal gas this can be written as

   $$p = p_0(A)\exp\left(-\frac{\Psi - \Psi_0}{RT}\right). \tag{3.87}$$

   This is the usual barometric formula with different base pressure $p_0(A)$ for each field line !

(b) Polytropic equation of state with $p = K\rho^\gamma$ with K and γ constant. We then have

$$\frac{\partial p}{\partial \psi} = -\left(\frac{p}{K}\right)^{1/\gamma} \tag{3.88}$$

leading to

$$\int_{p_0(A)}^{p} \left(\frac{p'}{K}\right)^{-1/\gamma} dp' = -(\Psi - \Psi_0). \tag{3.89}$$

Evaluating the integral we get

$$\frac{K\gamma}{\gamma - 1}\left[\left(\frac{p}{K}\right)^{\frac{\gamma-1}{\gamma}} - \left(\frac{p_0(A)}{K}\right)^{\frac{\gamma-1}{\gamma}}\right] = -(\Psi - \Psi_0). \tag{3.90}$$

Solving for p, we first obtain

$$\left(\frac{p}{K}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{p_0(A)}{K}\right)^{\frac{\gamma-1}{\gamma}} - \frac{\gamma-1}{\gamma}\frac{\Psi - \Psi_0}{K}. \tag{3.91}$$

We finally arrive at

$$p(A, \Psi) = K\left[\left(\frac{p_0(A)}{K}\right)^{\frac{\gamma-1}{\gamma}} - \frac{\gamma-1}{\gamma}\frac{\Psi - \Psi_0}{K}\right]^{\frac{\gamma}{\gamma-1}}. \tag{3.92}$$

We notice that the expression inside the bracket could become negative which will lead to mathematical and physical difficulties.

c) Use an equation of state plus an energy equation. This is the most difficult case, but also the most realistic one. We can again regard T as a function of A and Ψ, but this time unknown to us until we solve the energy equation.

3.4.2 Centrifugal force in cylindrical coordinates

We now turn to the case of a rotating plasma. This case is important e.g. for planetary magnetospheres likes those of Jupiter or the magnetospheres of rotating stars. It can also be of importance in laboratory plasmas which are heated by neutral beams, because the momentum transfer from the beam to the plasma can cause the plasma to rotate.

A rotating plasma violates our static assumption $\mathbf{v} = \mathbf{0}$, so in a strict sense we are not dealing with an MHS equilibrium. However, since the effect of pure steady rotation is simply to introduce an extra force, namely the centrifugal force we will include this case here.

Since $\mathbf{v} = \mathbf{0}$ we have to go back to the original MHD equations with $\mathbf{v}$ non-zero but $\partial/\partial t = 0$. We will work in cylindrical coordinates ϖ, ϕ, z, assume rotational invariance ($\partial/\partial\phi = 0$) and purely rotational motion around the z-axis ($\mathbf{v} = \varpi\Omega\mathbf{e}_\phi$).

The MHD equations then have the following form.

a) Continuity equation

$$\nabla \cdot (\rho \mathbf{v}) = \frac{1}{\varpi} \frac{\partial}{\partial \phi} (\varpi \rho \Omega) = 0 \tag{3.93}$$

because of axisymmetry.

b) Ohm's law and Faraday's law

From Faraday's law

$$\nabla \times \mathbf{E} = \mathbf{0} \tag{3.94}$$

we conclude that

$$\mathbf{E} = -\nabla \Phi. \tag{3.95}$$

With

$$\mathbf{B} = \frac{1}{\varpi} \nabla A \times \mathbf{e}_\phi + B_\phi \mathbf{e}_\phi \tag{3.96}$$

(from $\nabla \cdot \mathbf{B} = 0$) we find that

$$\begin{aligned} \mathbf{v} \times \mathbf{B} &= \varpi \Omega \mathbf{e}_\phi \times \left(\frac{1}{\varpi} \nabla A \times \mathbf{e}_\phi + B_\phi \mathbf{e}_\phi \right) \\ &= \Omega \nabla A \end{aligned} \tag{3.97}$$

so that the ideal Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0} \tag{3.98}$$

acquires the form

$$-\nabla \Phi + \Omega \nabla A = \mathbf{0}. \tag{3.99}$$

Taking the curl of this equation results in

$$\nabla \Omega \times \nabla A = \mathbf{0} \tag{3.100}$$

leading to the conclusion that

$$\Omega = H(A) \tag{3.101}$$

i.e. the angular velocity is constant along field lines (Ferraro's law of isorotation). Since Ω is a function of A, we also get

$$-\nabla \Phi + \Omega(A) \nabla A = \mathbf{0} \tag{3.102}$$

and conclude that the electric potential Φ is also a function of A ($\Phi = K(A)$) implying that the electric potential is constant along field lines.

c) Momentum balance equation

The momentum balance equation has the form

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{j}\times\mathbf{B} - \nabla p \qquad \overbrace{(-\rho\nabla\Psi)}^{\text{external gravity term could be included}} \quad . \tag{3.103}$$

The aim now is to bring the **v**-dependent term into the form $\rho\nabla\eta$ (with the centrifugal potential η). The most transparent way to do this is to write

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla\left(\frac{1}{2}|\mathbf{v}|^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v}). \tag{3.104}$$

The first term already has the form we want, so we only need to have a look at the second term. We get

$$\begin{aligned}\nabla\times\mathbf{v} &= \nabla\times(\varpi\Omega\mathbf{e}_\phi)\\ &= -\varpi\frac{\partial\Omega}{\partial z}\mathbf{e}_\varpi + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi^2\Omega)\mathbf{e}_z.\end{aligned} \tag{3.105}$$

and

$$\begin{aligned}\mathbf{v}\times(\nabla\times\mathbf{v}) &= \varpi\Omega\mathbf{e}_\phi\times\left(-\varpi\frac{\partial\Omega}{\partial z}\mathbf{e}_\varpi + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi^2\Omega)\mathbf{e}_z\right)\\ &= \varpi^2\Omega\frac{\partial\Omega}{\partial z}\mathbf{e}_z + \Omega\frac{\partial}{\partial\varpi}(\varpi^2\Omega)\mathbf{e}_\varpi\\ &= \nabla(\varpi^2\Omega^2) - \varpi^2\Omega\frac{d\Omega}{dA}\nabla A.\end{aligned} \tag{3.106}$$

Putting everything together, we get

$$\begin{aligned}&\frac{1}{2}\rho\nabla(\varpi^2\Omega^2) - \rho\nabla(\varpi^2\Omega^2) + \rho\varpi^2\Omega\frac{d\Omega}{dA}\nabla A =\\ &\quad\left[-\frac{1}{\mu_0}\nabla\cdot\left(\frac{1}{\varpi^2}\nabla A\right) - \frac{1}{\mu_0\varpi^2}b_\phi\frac{db_\phi}{dA}\right]\nabla A - \nabla p\end{aligned} \tag{3.107}$$

and finally

$$\left[-\frac{1}{\mu_0}\nabla\cdot\left(\frac{1}{\varpi^2}\nabla A\right) - \frac{1}{\mu_0\varpi^2}b_\phi\frac{db_\phi}{dA} - \rho\varpi^2\Omega\frac{d\Omega}{dA}\right]\nabla A - \nabla p + \rho\nabla\eta = \mathbf{0} \tag{3.108}$$

with $\eta = \varpi^2\Omega^2/2$ the centrifugal potential.

With the same arguments as before we conclude that

$$p = F(A,\eta) \tag{3.109}$$

and the partial differential equations to solve are

$$-\nabla\cdot\left(\frac{1}{\varpi^2}\nabla A\right) = \mu_0\left(\frac{\partial p}{\partial A}\right)_\eta + \frac{1}{\varpi^2}b_\phi\frac{db_\phi}{dA} + \mu_0\varpi^2\rho\Omega\frac{d\Omega}{dA} \tag{3.110}$$

$$\left(\frac{\partial p}{\partial\eta}\right)_A = \rho. \tag{3.111}$$

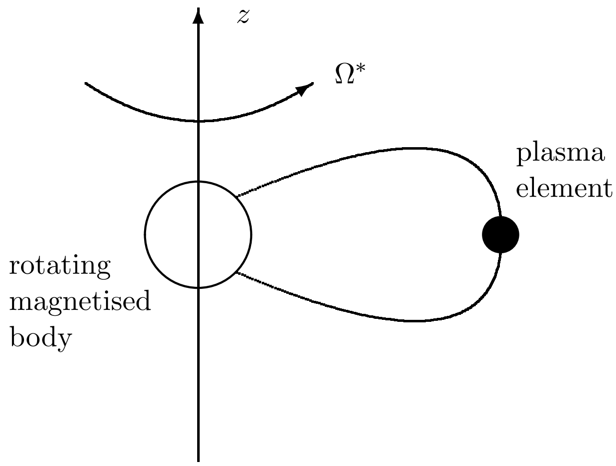

Figure 3.2: Illustration of Ferraro's law of isorotation

Again we have to provide information on ρ in the same way as before and to integrate the second equation first. When substituting p into the first equation one has to keep η constant although Ω can depend on A !

Ferraro's law of isorotation usually restricts Ω considerably (e.g. Figure 3.2). Imagine a rigidly rotating magnetised body, e.g. a star or a planet, which has a surface into which the field lines are frozen, i.e. Ω is fixed at the surface of the star. Then by Ferraro's law we have $\Omega(A) = \Omega^*$ for every field line touching the surface in at least one point. This can cause problems for field lines extending very far out because $v_\phi = \varpi\Omega^*$ will become very large. Of course this means that the centrifugal force becomes large and the plasma will be accelerated outwards: a plasma flow along the the field lines starts leading e.g. to a stellar wind and the field lines become open.

3.5 Some Useful Solutions for Symmetric Systems

We have now derived the fundamental elliptic PDE's for the three different cases of symmetric systems. We can use these equations to calculate equilibria. Before we actually do this some general remarks are necessary.

a) The equations still contain the unspecified functions $p(A)$ and/or $B_y(A)$ ($b_\phi(A)$, $h(A)$). We have to make choices which are sensible from the physical point of view. This is usually in conflict with mathematical

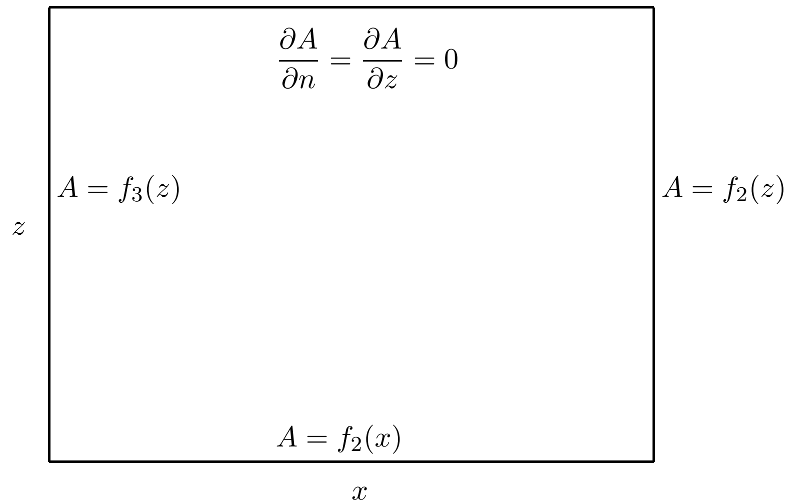

Figure 3.3: Example of boundary conditions in Cartesian geometry

simplicity which would require e.g. to make the PDE linear. It is for example obvious that $p(A)$ and $B_y^2(A)$ ($b_\phi^2(A)$, $h^2(A)$) have to be positive. This is not always ensured by some choices for p and $B_y^2(A)$ ($b_\phi^2(A)$, $h^2(A)$). In this part of the lecture we shall assume somewhat naively that we can prescribe p and B_y as we like subject to the constraints mentioned above.

b) We have to specify boundary conditions to complete the mathematical problem. However, usually people take the approach to try to find <u>any</u> solution of one of the PDE's first and tailor the boundary conditions for this solution afterwards. We will basically follow this approach, because otherwise it is hard to find any solutions, at least if we have a non-linear equation.

Since the equations are elliptic, the mathematically appropriate boundary conditions are e.g.

- Dirichlet boundary conditions: $A|_{boundary}$ =known function of space on boundary

  From a physical point of view this boundary condition fixes the point where the field line labeled 'A' crosses the boundary (sometimes called 'the footpoint'). Then, specifying A on the boundary is the same as specifying the magnetic flux distribution through that boundary !

  If $p(A)$ and $B_y(A)$ are prescribed, this also fixes p and B_y on the boundary ! This may not always be consistent with the physics of the problem studied ! I will get back to this point at the end of this Section.

- (Homogeneous) von Neumann boundary conditions (if imposed on all boundaries, an extra condition is necessary to make the solution unique): $\dfrac{\partial A}{\partial n}$ = known function of space, e.g. 0.
  This determines the angle with which the field line A intersects the boundary because

$$|\mathbf{B} \times \mathbf{n}| = |(\mathbf{n} \cdot \nabla A)\mathbf{e}_y| = |\mathbf{B}| \sin\theta \tag{3.112}$$

  where θ is the angle between $\mathbf{B}$ and $\mathbf{n}$. If, for example

$$\frac{\partial A}{\partial n} = 0 \quad \Longrightarrow \quad \mathbf{B} \parallel \mathbf{n} \tag{3.113}$$

  then $\mathbf{B}$ is perpendicular to the boundary.

Before I go on to discuss specific equilibria, I would like to introduce some widely used terminology.

a) Potential fields

Here $\mathbf{j} = \mathbf{0}$ and we have

$$\nabla \times \mathbf{B} = \mathbf{0}. \tag{3.114}$$

It follows that

$$\mathbf{B} = \nabla\varphi \tag{3.115}$$

and therefore

$$\nabla \cdot \mathbf{B} = \Delta\varphi = 0. \tag{3.116}$$

Equation (3.116) is the reason for the name potential fields. Note that φ is *not* identical with A in the symmetric cases, but if $\mathbf{j} = \mathbf{0}$ we have

$$-\Delta_2 A = 0 \quad \text{(translational symmetry)}, \tag{3.117}$$

$$-\nabla \cdot \left(\frac{1}{\varpi^2}\nabla A\right) = 0 \quad \text{(rotational symmetry)}, \tag{3.118}$$

$$-\nabla \cdot \left(\frac{1}{n^2 + k^2\varpi^2}\nabla A\right) = 0 \quad \text{(helical symmetry)}. \tag{3.119}$$

b) Force-free fields

This means that

$$\mathbf{j} \times \mathbf{B} = \mathbf{0}, \tag{3.120}$$

so the other forces are negligible. It follows that

$$\mu_0 \mathbf{j} = \alpha \mathbf{B}, \tag{3.121}$$

i.e. the current density is purely field aligned. Since

$$\nabla \cdot \mathbf{j} = \frac{1}{\mu_0} \nabla \cdot (\nabla \times \mathbf{B}) = 0, \tag{3.122}$$

we get

$$\mu_0 \nabla \cdot \mathbf{j} = \nabla \cdot (\alpha \mathbf{B}) = \mathbf{B} \cdot \nabla \alpha + \alpha \nabla \cdot \mathbf{B} = \mathbf{B} \cdot \nabla \alpha = 0, \tag{3.123}$$

implying that α is constant along magnetic field lines. If $\alpha =$ constant everywhere we have the so-called *constant-α force-free fields* or *linear force-free fields*, because as we will see, the equation determining $\mathbf{B}$ is linear in this case.

3.5.1 Solutions for Translational Invariance

a) $\underline{\mu_0 j_y = 0 \text{ solutions}}$

These include potential solutions, but in the case of translational symmetry without external forces we could also have

$$j_y = \frac{\partial}{\partial A} \left(p(A) + \frac{B_y^2(A)}{2\mu_0} \right) = 0 \tag{3.124}$$

implying

$$p(A) + \frac{B_y^2(A)}{2\mu_0} = c^2 = \text{constant}. \tag{3.125}$$

We can use this equation to determine either $p(A)$ or $B_y(A)$ if the other one is given, e.g.

$$B_y(A) = \pm \sqrt{2\mu_0 (c^2 - p(A))}, \tag{3.126}$$

where it must be ensured that $c^2 \geq p(A)$ for all A ! In this special case we have $j_y = 0$ but the the other components of the current density are non-zero

$$\begin{aligned} \mu_0 \mathbf{j} &= \nabla \times \mathbf{B} \\ &= \underbrace{-\Delta A}_{=0} \mathbf{e}_y + \nabla B_y \times \mathbf{e}_y \\ &= \frac{dB_y}{dA} \nabla A \times \mathbf{e}_y \neq \mathbf{0}. \end{aligned} \tag{3.127}$$

This special case does not exist if we have external forces or in the case of all other symmetries because then either p depends on the external potential ψ or we have extra explicit dependencies on the coordinates on the right hand side of the Grad-Shafranov equation.

The techniques for the solution, however, and the expressions for A are the same whatever case we have. We can use either separation of variables ($A = g(x)h(z)$ in Cartesian coordinates or $A = g(r)h(\phi)$ in polar coordinates) or complex analytic functions ($A = \Re[f(x+iz)]$) or Green's function techniques. All this is standard and I will not show any examples here.

b) $\mu_0 j_y = c$ =constant solutions

This is another case for which the Grad-Shafranov equation is linear, namely

$$-\Delta A = c \tag{3.128}$$

General solutions can be constructed by adding a potential solution A_0 with $-\Delta A_0 = 0$, to a particular solution of Eq. (3.128). An example would be

$$A = A_0 - \frac{1}{2}cx^2. \tag{3.129}$$

Since

$$\mu_0 p(A) + \frac{B_y^2}{2\mu_0} = cA + d \tag{3.130}$$

the solutions do only make sense if $cA + d \geq 0$.

c) $\mu_0 j_y = k^2 A$

This form of j_y includes linear force-free fields. The Grad-Shafranov equation has the form

$$-\Delta A = k^2 A = \mu_0 \frac{d}{dA}\left(p(A) + \frac{B_y^2(A)}{2\mu_0}\right). \tag{3.131}$$

We get linear force-free fields if p = constant, i.e.

$$-\Delta A = B_y \frac{dB_y}{dA} = k^2 A. \tag{3.132}$$

It follows that

$$B_y^2 = k^2 A^2 + B_{y0}^2. \tag{3.133}$$

Now

$$\begin{aligned}
\mu_0 \mathbf{j} &= \nabla B_y \times \mathbf{e}_y - \Delta A \, \mathbf{e}_y \\
&= \frac{dB_y}{dA} \nabla A \times \mathbf{e}_y + k^2 \, A \, \mathbf{e}_y \\
&= \frac{dB_y}{dA} \nabla A \times \mathbf{e}_y + B_y \frac{dB_y}{dA} \, \mathbf{e}_y \\
&= \frac{dB_y}{dA} \, \mathbf{B}.
\end{aligned} \tag{3.134}$$

If $B_{y0}^2 = 0$, we have $B_y = \pm|k|A$ and $dB_y/dA = \pm|k| = \alpha$!

Solutions can be obtained for example by separation of variables. In Cartesian coordinates we get

$$A = g(x) h(z) \tag{3.135}$$

leading to

$$-h \frac{d^2 g}{dx^2} - g \frac{d^2 h}{dz^2} = k^2 \, g \, h \tag{3.136}$$

when substituted into Eq. (3.132). Dividing by $g \, h$ we get

$$-\frac{1}{g} \frac{d^2 g}{dx^2} = k^2 + \frac{1}{h} \frac{d^2 h}{dz^2} = c^2 \tag{3.137}$$

with c^2 constant. If we choose c^2 to be positive the solutions are

$$g = g_1 \sin(cx) + g_2 \cos(cx) \tag{3.138}$$

$$h = h_1 \exp(\sqrt{c^2 - k^2} z) + h_2 \exp(-\sqrt{c^2 - k^2} z) \tag{3.139}$$

Here g_1, g_2, h_1 and h_2 are constants. In the case of linear force-free fields, we can replace k^2 by α^2. For all linear equations we can superpose different solutions to match boundary conditions for example. That is one of the reasons why the linear Grad-Shafranov equations are so popular.

d) $\underline{\mu_0 j_y = \lambda \exp(2 \, A)}$

This is the first non-linear current density we investigate and is a very popular choice for two reasons:

(a) the complete set of solutions is known explicitly (Liouville, 1853) and the equation has particularly nice properties as conformal invariance;

(b) there is a physical justification for this current profile (!), because it results from a kinetic approach with Maxwellian distribution functions where the particles drift in the y-direction and the plasma is quasi-neutral. The plasma is then in local thermodynamic equilibrium. This argument does only apply, however, to a j_y caused by a pressure gradient and not to magnetic shear !

So the Grad-Shafranov equation is

$$-\Delta A = \lambda \exp(2\,A). \tag{3.140}$$

This equation is sometimes called Liouville's equation. The solutions to this equation are given by

$$A = -\ln\left(\frac{1+\frac{1}{4}\lambda|\psi(u)|^2}{\left|\frac{d\psi}{du}\right|}\right) \tag{3.141}$$

where $u = x + \mathrm{i}z$ and ψ is an analytic function, or written in a slightly different way

$$\lambda \exp(2A) = \frac{\lambda\left|\frac{d\psi}{du}\right|}{\left(1+\frac{1}{4}\lambda|\psi(u)|^2\right)^2}. \tag{3.142}$$

Notice that for $\lambda \to 0$, we have $j_y \to 0$ and

$$A \to \ln\left(\left|\frac{d\psi}{du}\right|\right), \tag{3.143}$$

which is a harmonic function !

Let us first verify that A does indeed solve the PDE. It is relatively straightforward to to this if we write

$$\psi(u) = g(x,z) + \mathrm{i}h(x,z) \tag{3.144}$$

with g and h satisfying the Cauchy-Riemann equations for harmonic functions

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial z} \tag{3.145}$$

$$\frac{\partial g}{\partial z} = -\frac{\partial h}{\partial x}. \tag{3.146}$$

Then

$$|\psi|^2 = g^2 + h^2 \tag{3.147}$$

and

$$\left|\frac{d\psi}{du}\right| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2} = |\nabla g|. \tag{3.148}$$

We can therefore write A as

$$\begin{aligned} A &= -\ln\left(\frac{1+\frac{1}{4}\lambda(g^2+h^2)}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2}}\right) \\ &= -\ln\left(1+\frac{1}{4}\lambda(g^2+h^2)\right) + \frac{1}{2}\ln\left[\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2\right]. \end{aligned} \tag{3.149}$$

Now the complex analytic function

$$\ln\left(\frac{d\psi}{du}\right) = \ln\left|\frac{d\psi}{du}\right| + \mathrm{i}\left\{\arctan\left[\frac{\Im\left(\frac{d\psi}{du}\right)}{\Re\left(\frac{d\psi}{du}\right)}\right] + 2k\pi\right\} \tag{3.150}$$

and this implies that

$$-\Delta\ln\left|\frac{d\psi}{du}\right| = -\Delta\frac{1}{2}\ln\left[\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2\right] = 0. \tag{3.151}$$

So we only have to calculate the effect of Laplace's operator onto the first term given by

$$Q = -\ln\left(1+\frac{1}{4}\lambda(g^2+h^2)\right). \tag{3.152}$$

We get

$$\frac{\partial Q}{\partial x} = -\frac{\frac{1}{2}\lambda\left(g\frac{\partial g}{\partial x} + h\frac{\partial h}{\partial x}\right)}{1+\frac{1}{4}\lambda(g^2+h^2)} \tag{3.153}$$

$$\frac{\partial Q}{\partial z} = -\frac{\frac{1}{2}\lambda\left(g\frac{\partial g}{\partial z} + h\frac{\partial h}{\partial z}\right)}{1+\frac{1}{4}\lambda(g^2+h^2)}. \tag{3.154}$$

For the second derivatives we obtain

$$\frac{\partial^2 Q}{\partial x^2} = -\frac{1}{\left[1+\frac{1}{4}\lambda(g^2+h^2)\right]^2}.$$

$$\left\{\frac{1}{2}\lambda\left[\left(\frac{\partial g}{\partial x}\right)^2+g\frac{\partial^2 g}{\partial x^2}+\left(\frac{\partial h}{\partial x}\right)^2+h\frac{\partial^2 h}{\partial x^2}\right]\left[1+\frac{1}{4}\lambda(g^2+h^2)\right]\right.$$
$$\left.-\frac{1}{4}\lambda^2\left(g\frac{\partial g}{\partial x}+h\frac{\partial h}{\partial x}\right)^2\right\} \tag{3.155}$$

$$\frac{\partial^2 Q}{\partial z^2}=-\frac{1}{\left[1+\frac{1}{4}\lambda(g^2+h^2)\right]^2}\cdot$$
$$\left\{\frac{1}{2}\lambda\left[\left(\frac{\partial g}{\partial z}\right)^2+g\frac{\partial^2 g}{\partial z^2}+\left(\frac{\partial h}{\partial z}\right)^2+h\frac{\partial^2 h}{\partial z^2}\right]\left[1+\frac{1}{4}\lambda(g^2+h^2)\right]\right.$$
$$\left.-\frac{1}{4}\lambda^2\left(g\frac{\partial g}{\partial z}+h\frac{\partial h}{\partial z}\right)^2\right\}. \tag{3.156}$$

Putting everything together, we obtain

$$\begin{aligned}-\Delta A &= -\frac{\partial^2 Q}{\partial x^2}-\frac{\partial^2 Q}{\partial z^2}\\ &= \frac{1}{\left[1+\frac{1}{4}\lambda(g^2+h^2)\right]^2}\cdot\end{aligned}$$
$$\left\{\frac{1}{2}\lambda\left[\left(\frac{\partial g}{\partial x}\right)^2+g\frac{\partial^2 g}{\partial x^2}+\left(\frac{\partial h}{\partial x}\right)^2+h\frac{\partial^2 h}{\partial x^2}\right]\left[1+\frac{1}{4}\lambda(g^2+h^2)\right]+\right.$$
$$\frac{1}{2}\lambda\left[\left(\frac{\partial g}{\partial z}\right)^2+g\frac{\partial^2 g}{\partial z^2}+\left(\frac{\partial h}{\partial z}\right)^2+h\frac{\partial^2 h}{\partial z^2}\right]\left[1+\frac{1}{4}\lambda(g^2+h^2)\right]-$$
$$\left.\frac{1}{4}\lambda^2\left(g\frac{\partial g}{\partial x}+h\frac{\partial h}{\partial x}\right)^2-\frac{1}{4}\lambda^2\left(g\frac{\partial g}{\partial z}+h\frac{\partial h}{\partial z}\right)^2\right\}. \tag{3.157}$$

Since

$$\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial z^2}=\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial z^2}=0, \tag{3.158}$$

we get

$$-\Delta A=\frac{1}{\left[1+\frac{1}{4}\lambda(g^2+h^2)\right]^2}\cdot$$
$$\left\{\frac{1}{2}\lambda\left[\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial z}\right)^2+\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial z}\right)^2\right]\left[1+\frac{1}{4}\lambda(g^2+h^2)\right]\right.$$
$$-\frac{1}{4}\lambda^2\left[g^2\left(\frac{\partial g}{\partial x}\right)^2+2gh\frac{\partial g}{\partial x}\frac{\partial h}{\partial x}+h^2\left(\frac{\partial h}{\partial x}\right)^2+\right.$$
$$\left.\left.g^2\left(\frac{\partial g}{\partial z}\right)^2+2gh\frac{\partial g}{\partial z}\frac{\partial h}{\partial z}+h^2\left(\frac{\partial h}{\partial z}\right)^2\right]\right\}. \tag{3.159}$$

Using the Cauchy-Riemann equations we finally obtain

$$
\begin{aligned}
-\Delta A = & \frac{1}{\left[1+\frac{1}{4}\lambda(g^2+h^2)\right]^2} \cdot \\
& \left\{ \lambda \left[\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2 \right] \left[1+\frac{1}{4}\lambda(g^2+h^2)\right] \right. \\
& -\frac{1}{4}\lambda^2(g^2+h^2) \left[\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2 \right] \\
& \left. -\frac{1}{4}\lambda^2\, 2gh \left[\underbrace{\frac{\partial g}{\partial x}\frac{\partial h}{\partial x} - \frac{\partial h}{\partial x}\frac{\partial g}{\partial x}}_{=0} \right] \right\} . \\
= & \lambda \frac{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2}{\left[1+\frac{1}{4}\lambda(g^2+h^2)\right]^2} \\
= & \lambda \exp(-2A), \qquad (3.160)
\end{aligned}
$$

thus completing the proof that Eq. (3.141) is actually a solution of Eq. (3.140).

We will now discuss some important special cases.

- In the first case we choose

  $$\psi = \frac{2}{\sqrt{\lambda}} \exp(\sqrt{\lambda} u) \qquad (3.161)$$

  implying

  $$|\psi| = \frac{2}{\sqrt{\lambda}} \exp(\sqrt{\lambda} x) \qquad (3.162)$$

  and

  $$\left|\frac{d\psi}{du}\right| = 2\exp(\sqrt{\lambda} x). \qquad (3.163)$$

  Substitution into Eq. (3.141) gives

  $$
  \begin{aligned}
  A &= -\ln \left(\frac{1+\frac{\lambda}{4}\frac{4}{\lambda}\exp(2\sqrt{\lambda}x)}{2\exp(\sqrt{\lambda}x)} \right) \\
  &= -\ln \left(\frac{\exp(\sqrt{\lambda}x)+\exp(-\sqrt{\lambda}x)}{2} \right) \\
  &= -\ln\cosh\left(\sqrt{\lambda}x\right). \qquad (3.164)
  \end{aligned}
  $$

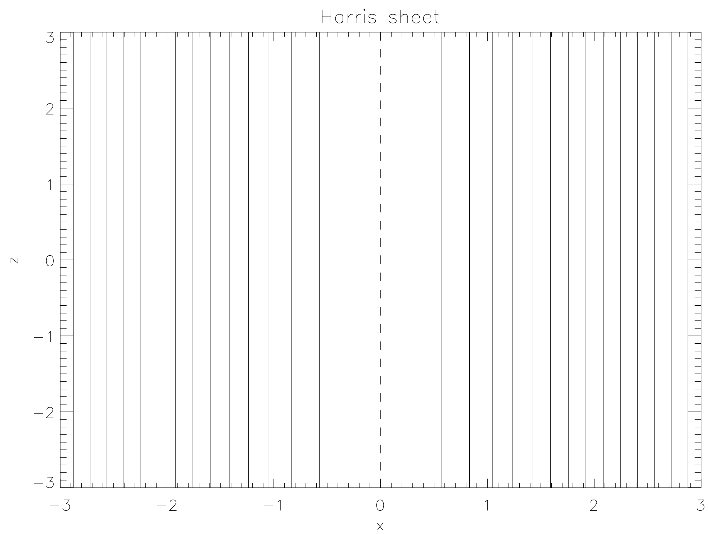

Figure 3.4: Field lines for the Harris sheet solution.

In the case $B_y = 0$ this solution is called either *Harris sheet* (Harris, 1962) or *(plane) sheet pinch*. In this case we have

$$\begin{aligned} \mathbf{B} &= \nabla A \times \mathbf{e}_y \\ &= -\sqrt{\lambda}\tanh(\sqrt{\lambda}x), \end{aligned} \quad (3.165)$$

$$j_y = \frac{\lambda}{\cosh^2(\sqrt{\lambda}x)} \quad (3.166)$$

$$p = p_0 + \frac{\lambda}{2}\frac{1}{\cosh^2(\sqrt{\lambda}x)}. \quad (3.167)$$

As $|x| \to \infty$ the magnetic field tends to a constant field

$$\lim_{|x|\to\infty} |\mathbf{B}| = \sqrt{\lambda}, \quad (3.168)$$

but the magnetic field lines point into opposite directions for positive and negative x. A field line plot is shown in Fig. 3.4.

- Another important case in Cartesian coordinates is given by

$$\psi = \frac{2}{\sqrt{\lambda}}\left[\delta + (1+\delta^2)^{1/2}\exp(\sqrt{\lambda}u)\right] \quad (3.169)$$

$$\begin{aligned} |\psi|^2 &= \frac{4}{\lambda}\left[\delta^2 + 2\delta(1+\delta^2)^{1/2}\exp(\sqrt{\lambda}x)\cos(\sqrt{\lambda}z)\right. \\ &\qquad \left. +(1+\delta^2)\exp(2\sqrt{\lambda}x)\right] \end{aligned} \quad (3.170)$$

$$\left|\frac{d\psi}{du}\right| = 2(1+\delta^2)^{1/2}\exp(2\sqrt{\lambda}x) \quad (3.171)$$

Substitution into Eq. (3.141) results in

$$A = - \ln \left\{ \frac{1}{2(1+\delta^2)^{1/2}\exp(2\sqrt{\lambda}x)} \cdot \left[1+\delta^2 + 2\delta(1+\delta^2)^{1/2}\exp(\sqrt{\lambda}x)\cos(\sqrt{\lambda}z) + (1+\delta^2)\exp(2\sqrt{\lambda}x) \right] \right\}$$

$$= - \ln \left\{ (1+\delta^2)^{1/2} \cdot \left[\cosh(\sqrt{\lambda}x) + \frac{\delta}{(1+\delta^2)^{1/2}} \cos(\sqrt{\lambda}z) \right] \right\}. \quad (3.172)$$

In the limit $\delta \to 0$, we recover the Harris sheet. In the limit $\delta \to \infty$, the factor of the cosine becomes equal to one and the argument of the logarithm is $\cosh(\sqrt{\lambda}x) + \cos(\sqrt{\lambda}z)$, which becomes zero for $x = 0$ and $\sqrt{\lambda}z = k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$.

For the current density j_y we get

$$j_y = \frac{\lambda}{(1+\delta^2)\left[\cosh(\sqrt{\lambda}x) + \frac{\delta}{(1+\delta^2)^{1/2}} \cos(\sqrt{\lambda}z) \right]^2}, \quad (3.173)$$

so $j_y \to \infty$ for $\delta \to \infty$ and $x = 0$ and $\sqrt{\lambda}z = k\pi$ with $k = 0, \pm 1, \pm 2, \ldots$.

This solution has been first discussed by Fadeev et al. (1965) and Schmid-Burgk (1965). It is called *periodic* or *corrugated sheet pinch* (sometimes also *Schmid-Burgk solution*). A plot of the field line structure is shown in Fig. 3.5.

- Another important case, best discussed in polar coordinates, is the following

$$\psi = 2ku \quad (3.174)$$

$$|\psi| = 2k\varpi \qquad (\varpi = |u|) \quad (3.175)$$

$$\left| \frac{d\psi}{du} \right| = 2k. \quad (3.176)$$

Using these expressions in Eq. (3.141) leads to

$$A = - \ln \left(\frac{1 + \lambda k^2 \varpi^2}{2k} \right). \quad (3.177)$$

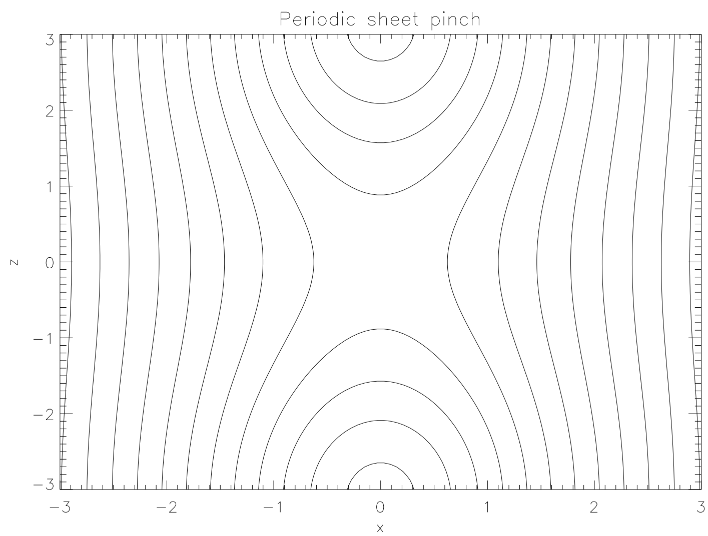

Figure 3.5: Field lines for the periodic sheet pinch solution.

For the case $B_y = 0$, $p = \lambda \exp(2A)$ this solution is called the *Bennett pinch* (sometimes also called the *peaked current profile*) (Bennett, 1934). This is an example of a so-called *z-pinch* for which $\mathbf{j} = j\mathbf{e}_z$ (though here it should be called a *y*-pinch). A plot of the field line structure is shown in Fig. 3.6. In the case of the corresponding force-free solution ($p =$ constant, $B_y = \sqrt{\lambda}\exp(A)$), we obtain the so-called *Gold-Hoyle solution* (Gold and Hoyle, 1960) with (ϖ, ϕ, y) is the slightly awkward coordinate system now)

$$B_\phi = (\nabla A \times \mathbf{e}_y) \cdot \mathbf{e}_\phi = \frac{\frac{1}{2}\lambda k^2 \varpi}{1 + \frac{1}{4}k^2\varpi^2} \quad (3.178)$$

$$B_y = \frac{\sqrt{\lambda}k}{1 + \frac{1}{4}k^2\varpi^2}. \quad (3.179)$$

Note that the Bennett pinch and the Gold-Hoyle solution depend only on ϖ. Therefore they are both translationally and rotationally invariant and are also solutions of the Grad-Shafranov equation for rotational invariance ! But notice that then $A_{rot} \neq A_{cart}$, and $B_y^{cart} = B_z^{rot} = 1/\varpi \nabla A_{rot} \times \mathbf{e}_\phi$, whereas $b_\phi^{rot}(A_{rot}) = \varpi B_\phi^{cart}$!

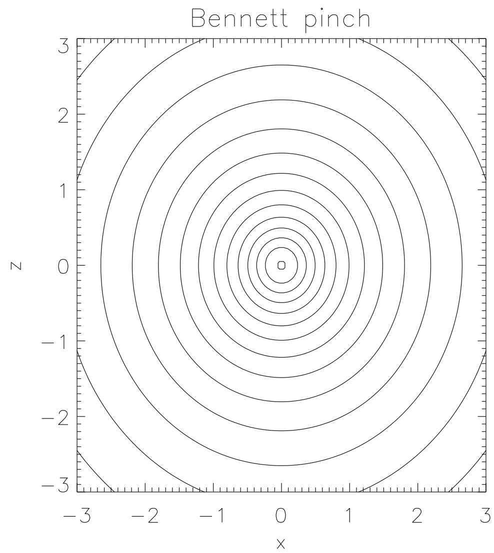

Figure 3.6: Field lines for the Bennett pinch solution.

3.5.2 Solutions for Rotational Invariance

a) $\underline{\mu_0 j_\phi = 0}$

In the case of rotational invariance these solutions are potential because

$$\mu_0 j_\phi = \mu_0 \varpi \frac{dp}{dA} + \frac{1}{\varpi} b_\phi \frac{db_\phi}{dA} \tag{3.180}$$

and this cannot vanish except dp/dA and db_ϕ/dA vanish separately. The equation to solve is

$$-\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\frac{1}{\varpi}\frac{\partial A}{\partial\varpi}\right) - \frac{1}{\varpi^2}\frac{\partial^2 A}{\partial z^2} = 0. \tag{3.181}$$

This is *not* the Laplace equation in cylindrical coordinates (ignoring the ϕ derivatives) !

We can solve this equation by separation of variables. We first transform the radial coordinate by

$$R = \frac{1}{2}\varpi^2. \tag{3.182}$$

Equation (3.181) can the be written as

$$2R\frac{\partial^2 A}{\partial R^2} + \frac{\partial^2 A}{\partial z^2} = 0. \tag{3.183}$$

Now let

$$A = g(R)h(z) \tag{3.184}$$

and we get

$$\frac{2R}{g}\frac{d^2g}{dR^2} = -\frac{1}{h}\frac{d^2h}{dz^2} = -c^2 = \text{constant}. \tag{3.185}$$

It follows that

$$\begin{aligned} \frac{d^2g}{dR^2} + \frac{c^2}{2R}g &= 0 & (3.186)\\ \frac{d^2h}{dz^2} - c^2h &= 0. & (3.187) \end{aligned}$$

The solutions to these equations are given by

$$\begin{aligned} g &= \sqrt{R}\left[g_1 \mathrm{J}_1\left(c\sqrt{2R}\right) + g_2 \mathrm{Y}_1\left(c\sqrt{2R}\right)\right] \\ &= \sqrt{\frac{1}{2}}\varpi\left[g_1 \mathrm{J}_1(c\varpi) + g_2 \mathrm{Y}_1(c\varpi)\right] & (3.188)\\ h &= h_1 \exp(cz) + h_2 \exp(-cz) & (3.189) \end{aligned}$$

for $c^2 > 0$. Here g_1, g_2, h_1 and h_2 are constants and J_1 and Y_1 are Bessel functions. For $c^2 < 0$ we get

$$\begin{aligned} g &= \sqrt{\frac{1}{2}}\varpi\left[g_1 \mathrm{I}_1(|c|\varpi) + g_2 \mathrm{K}_1(|c|\varpi)\right] & (3.190)\\ h &= h_1 \sin(|c|z) + h_2 \cos(|c|z) & (3.191) \end{aligned}$$

with I_1 and K_1 being modified Bessel functions.

Of particular importance in astrophysics are the rotationally symmetric solutions in spherical coordinates r, θ and ϕ. Then

$$\mathbf{B} = \frac{1}{r\sin\theta}\nabla A \times \mathbf{e}_\phi + B_\phi \mathbf{e}_\phi \tag{3.192}$$

and Eq. (3.181) becomes

$$-\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A}{\partial r^2} - \frac{1}{r^4\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial A}{\partial\theta}\right) = 0. \tag{3.193}$$

If we introduce the new variable $\mu = \cos\theta$ we get

$$-r^2\frac{\partial^2 A}{\partial r^2} - (1-\mu^2)\frac{\partial^2 A}{\partial \mu^2} = 0. \tag{3.194}$$

We solve this by separation of variables using

$$A = g(r)h(\mu) \qquad (= r \overbrace{\sin\theta}^{=\sqrt{1-\mu^2}} A_\phi) \tag{3.195}$$

and get

$$-\frac{r^2}{g}\frac{d^2 g}{dr^2} = (1-\mu^2)\frac{1}{h}\frac{d^2 h}{d\mu^2} = -l(l+1) \tag{3.196}$$

with l integer. The equations to solve are

$$\frac{d^2 g}{dr^2} - \frac{l(l+1)}{r^2} g = 0 \tag{3.197}$$

with the solution

$$g = g_1 r^{l+1} + g_2 r^{-l} \tag{3.198}$$

and

$$\frac{d^2 h}{d\mu^2} + \frac{l(l+1)}{1-\mu^2} h = 0 \tag{3.199}$$

with the solution

$$h = \sqrt{1-\mu^2}\left[h_1 \mathrm{P}_l^1(\mu) + h_2 \mathrm{Q}_l^1(\mu)\right]; \quad l \geq 1 \tag{3.200}$$

with $\mathrm{P}_l^m(x)$ and $\mathrm{Q}_l^m(x)$ being associated Legendre functions. The functions Q_l^m are singular at $x = \pm 1$ and therefore we assume $h_2 = 0$ from now on. The functions $\mathrm{P}_l^m(x)$ are polynomials defined by

$$\mathrm{P}_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} \mathrm{P}_l(x) \tag{3.201}$$

with $\mathrm{P}_l(x)$ the Legendre polynomial of degree l. In Eq. (3.200) $l = 0$ has been explicitly excluded. We can, however, study this case by solving Eqs. (3.197) and (3.199) directly for $l = 0$. The solutions are given by

$$g = g_1 r + g_2 \tag{3.202}$$

$$h = h_1 \mu + h_2 = h_1 \cos\theta + h_2. \tag{3.203}$$

We focus on the case $g_1 = h_2 = 0$ leading to

$$A = g_2 h_1 \cos\theta \tag{3.204}$$

giving

$$\begin{aligned} \mathbf{B} &= \frac{1}{r\sin\theta}\nabla A \times \mathbf{e}_\phi \\ &= \frac{1}{r\sin\theta}\left(-g_2 h_1 \frac{\sin\theta}{r}\mathbf{e}_\theta \times \mathbf{e}_\phi\right) \\ &= -\frac{g_2 h_1}{r^2}\mathbf{e}_r, \end{aligned} \tag{3.205}$$

the field of a magnetic monopole !

For $l = 1$ we have

$$\begin{aligned} \mathrm{P}_1 &= \mu = \cos\theta && (3.206)\\ \mathrm{P}_1^1 &= (-1)\sin\theta\frac{d}{d\mu}\mu = -\sin\theta. && (3.207) \end{aligned}$$

It follows that

$$h = h_1 \sin\theta \mathrm{P}_1^1 = -h_1 \sin^2\theta. \quad (3.208)$$

We first discuss the case with $g_2 = 0$. Then we have

$$A = -g_1 h_1 r^2 \sin^2\theta \qquad (= -g_1 h_1 \varpi^2 \quad !) \quad (3.209)$$

leading to

$$\begin{aligned} \mathbf{B} &= \frac{1}{r\sin\theta}\nabla A \times \mathbf{e}_\phi \\ &= -\frac{g_1 h_1}{r\sin\theta}\left(\frac{1}{r}2r^2\cos\theta\sin\theta\mathbf{e}_r - 2r\sin^2\theta\mathbf{e}_\theta\right) \\ &= -2g_1 h_1 \left(\underbrace{\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta}_{=\mathbf{e}_z!}\right), \end{aligned} \quad (3.210)$$

so the magnetic field is constant and pointing into the z direction. The case $g_1 = 0$ results in

$$A = -g_2 h_1 \frac{\sin^2\theta}{r} \quad (3.211)$$

and

$$\begin{aligned} \mathbf{B} &= -\frac{g_2 h_1}{r\sin\theta}\left(-\frac{2\cos\theta\sin\theta}{r^2}\mathbf{e}_r - \frac{\sin^2\theta}{r^2}\mathbf{e}_\theta\right) \\ &= -g_2 h_1 \left(-\frac{2\cos\theta}{r^3}\mathbf{e}_r - \frac{\sin\theta}{r^3}\mathbf{e}_\theta\right). \end{aligned} \quad (3.212)$$

To understand this expression better, we look at the vector potential of a magnetic dipole field with the dipole moment $\mathbf{m}$ located at the origin and pointing into the z direction. The vector potential is given by

$$\begin{aligned} \mathbf{A} &\propto \frac{\mathbf{m}\times\mathbf{x}}{r^3} \\ &\propto \frac{|\mathbf{m}|\varpi}{r^3}\mathbf{e}_\phi \\ &\propto \frac{|\mathbf{m}| r\sin\theta}{r^3}\mathbf{e}_\phi \\ &\propto \frac{|\mathbf{m}|\sin\theta}{r^2}\mathbf{e}_\phi. \end{aligned} \quad (3.213)$$

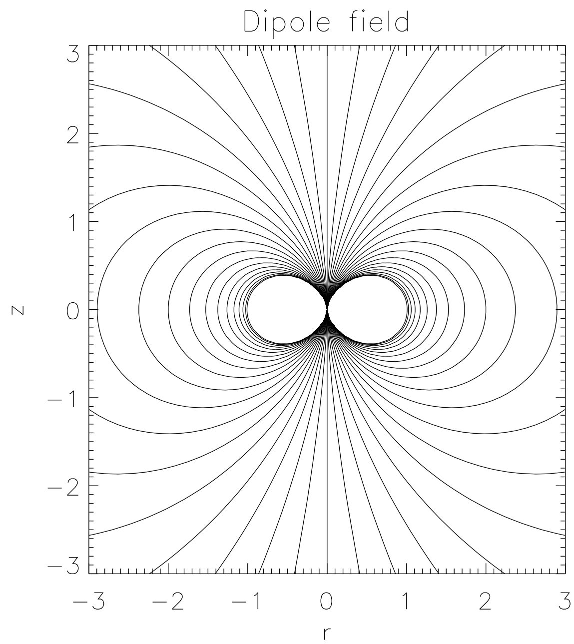

Figure 3.7: Field lines of the dipole solution.

Here $|\mathbf{x}| = r$ and $\mathbf{x} = \varpi \mathbf{e}_\varpi + z\mathbf{e}_z$. The vector potential is related to the flux function by

$$A = r \sin\theta A_\phi \propto |\mathbf{m}| \frac{\sin^2\theta}{r}. \tag{3.214}$$

It follows that the flux function given in Eq. (3.211) is that of a magnetic dipole field with the dipole moment aligned with the z-axis. A field line plot of that is shown in Fig. 3.7.

Higher values of l give higher order multipoles like quadrupole, octupole etc.

b) $\underline{\mu_0 j_\phi = \mu_0 \varpi p_1' + b_{\phi 1}^2 / 2\varpi}$

In this case the pressure is given by

$$p = p_1' A + p_0 \tag{3.215}$$

with p_1' and p_0 being constant and

$$b_\phi^2 = b_{\phi 1}^2{\prime} A + b_{\phi 0}^2. \tag{3.216}$$

This is the case which corresponds to a constant current density in translational symmetry, but here the current density is not constant. It is, however, independent of A.

We can again find solutions as superposition of potential solutions ($j_\phi = 0$) and a particular solution of the full equation.

One solution sometimes referred to in the fusion literature is *Solov'ev's solution* (Solov'ev, 1968). This solution is given in polynomial form as

$$A = f_1(\varpi) z^2 + f_2(\varpi). \tag{3.217}$$

Substituting this into the Grad-Shafranov equation we obtain

$$-\frac{1}{\varpi}\frac{d}{d\varpi}\left(\frac{1}{\varpi}\frac{df_1}{d\varpi}\right) z^2 - \frac{1}{\varpi}\frac{d}{d\varpi}\left(\frac{1}{\varpi}\frac{df_2}{d\varpi}\right) - \frac{2}{\varpi^2} f_1 = \mu_0 \varpi p_1' + \frac{1}{2\varpi} b_{\phi 1}^2. \tag{3.218}$$

By equating the coefficients of equal powers of z on the left and on the right hand side of this equation we get

$$\frac{d}{d\varpi}\left(\frac{1}{\varpi}\frac{df_1}{d\varpi}\right) = 0 \tag{3.219}$$

with the solution

$$f_1 = a_1 \varpi^2 + a_2 \varpi + a_3. \tag{3.220}$$

The equation for f_2 is given by

$$-\frac{1}{\varpi}\frac{d}{d\varpi}\left(\frac{1}{\varpi}\frac{df_2}{d\varpi}\right) = +\frac{2}{\varpi^2} f_1 + \mu_0 \varpi p_1' + \frac{1}{2\varpi} b_{\phi 1}^2. \tag{3.221}$$

This equation can be directly integrated using Eq. (3.220) and we get for f_2

$$\begin{aligned} f_2 = &-\frac{1}{4}\left(a_1 + \frac{1}{2}\mu_0 p_1'\right)\varpi^4 - \frac{2}{3} a_2 \varpi^3 \\ &+ \frac{1}{2} a_4 \varpi^2 - \left(2a_3 + \frac{1}{2} b_{\phi 1}^2\right)\left(\frac{\varpi^2}{2}\ln\varpi - \frac{\varpi^2}{4}\right) + a_5. \end{aligned} \tag{3.222}$$

We get Solov'ev's solution if we set

$$a_2 = 0 \tag{3.223}$$

$$a_3 = -\frac{1}{4} b_{\phi 1}^2. \tag{3.224}$$

The flux function A is then given by

$$\begin{aligned} A = &\, a_1\left(\varpi^2 - \frac{b_{\phi 1}^2}{4a_1}\right) z^2 \\ &- \frac{1}{4}\left(a_1 + \frac{1}{2}\mu_0 p_1'\right)\left(\varpi^4 - \frac{2a_4}{a_1 + \frac{1}{2}\mu_0 p_1'}\varpi^2 - \frac{a_5}{a_1 + \frac{1}{2}\mu_0 p_1'}\right). \end{aligned} \tag{3.225}$$

By further setting

$$\frac{a_4}{a_1 + \frac{1}{2}\mu_0 p_1'} = R_0^2 \tag{3.226}$$

$$\frac{a_5}{a_1 + \frac{1}{2}\mu_0 p_1'} = R_0^4 \tag{3.227}$$

$$a_1 = -\frac{\mu_0 p_1'}{2(1+\alpha^2)} \tag{3.228}$$

$$\frac{b_{\phi 1}^2}{4a_1} = \gamma \tag{3.229}$$

we finally arrive at

$$A = -\frac{\mu_0 p_1'}{2(1+\alpha^2)}\left[(\varpi^2 - \gamma)z^2 + \frac{\alpha^2}{4}(\varpi^2 - R_0^2)^2\right] \tag{3.230}$$

c) $\underline{\mu_0 j_\phi = \varpi k_1^2 A + k_2^2 A/\varpi}$

The Grad-Shafranov equation in this case is

$$-\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\frac{1}{\varpi}\frac{\partial A}{\partial\varpi}\right) - \frac{1}{\varpi^2}\frac{\partial^2 A}{\partial z^2} = k_1^2 A + \frac{1}{\varpi^2}k_2^2 A. \tag{3.231}$$

This particular case includes the axisymmetric linear force-free fields if $k_1 = 0$. We try separation of variables again and with

$$A = g(\varpi)h(z) \tag{3.232}$$

we obtain

$$-\frac{\varpi}{g}\frac{d}{d\varpi}\left(\frac{1}{\varpi}\frac{dg}{d\varpi}\right) - k_1^2\varpi^2 - k_2^2 = \frac{1}{h}\frac{d^2 h}{dz^2} = c^2. \tag{3.233}$$

For $c^2 >$ we have for h the solution

$$h = h_1\exp(cz) + h_2\exp(-cz). \tag{3.234}$$

and for $c^2 < 0$ one gets

$$h = h_1\sin(|c|z) + h_2\cos(|c|z). \tag{3.235}$$

The equation for g reads

$$\frac{1}{\varpi}\frac{d}{d\varpi}\left(\frac{1}{\varpi}\frac{dg}{d\varpi}\right) + \left(\frac{k_2^2 + c^2}{\varpi^2} + k_1^2\right) g = 0 \tag{3.236}$$

and with the transformation $R = \varpi^2/2$ becomes

$$\frac{d^2g}{dR^2} + \left(\frac{k_2^2 + c^2}{2R} + k_1^2\right) g = 0. \tag{3.237}$$

The solutions of this equation are called Coulomb wave functions with angular momentum number $L = 0$. These functions can be expressed in terms of confluent hypergeometric functions (or Kummer functions) (see Abramowitz and Stegun, 1965, , Chapter 14).

If $k_1 = 0$ (the force-free case) the (non-singular) solution for g is given by

$$g = g_1 \varpi \mathrm{J}_1(\sqrt{k_2^2 + c^2}\varpi). \tag{3.238}$$

If $c = 0$ as well, the corresponding magnetic field components are given by

$$\begin{aligned}
B_z &= g_1 \frac{1}{\varpi}\frac{\partial}{\partial \varpi}\left[\mathrm{J}_1(k_2\varpi)\right] \\
&= g_1 \left[\frac{1}{\varpi}\mathrm{J}_1(k_2\varpi) + k_2 \mathrm{J}_1'(k_2\varpi)\right] \\
&= g_1 \left[\frac{1}{\varpi}\mathrm{J}_1(k_2\varpi) + k_2 \mathrm{J}_0(k_2\varpi) - \frac{1}{\varpi}\mathrm{J}_1(k_2\varpi)\right] \\
&= g_1 k_2 \mathrm{J}_0(k_2\varpi) \qquad (3.239) \\
B_\phi &= \frac{b_\phi}{\varpi} = \frac{k_2^2}{\varpi} A = g_1 k_2^2 \mathrm{J}_1(k_2\varpi). \qquad (3.240)
\end{aligned}$$

This solution is sometimes called the *reversed field pinch* because B_z reverses its direction at the first zero of J_0.

For $c^2 > 0$ and $k_1 = 0$ one obtains another force-free solution which in solar physics is usually called the Schatzmann solution (Schatzmann, 1965). This solution has a similar structure for g, but exponentials for h. This solution was proposed as an early model of the magnetic fields of sunspots.

d) Nonlinear cases

Some 1D solutions have been found, e.g. an equivalent to the Harris sheet was discovered by Pfirsch (1962). Non-linear 2D solutions are very difficult to find in rotational symmetry.

3.5.3 Solutions for Helical Invariance

I will not discuss any helical equilibria here, because they are only rarely used in solar or astrophysical applications. Also, they are by no means easy to calculate even for the linear cases.

3.5.4 External Forces

If external forces are included the current density depends explicitly on the coordinates and it is usually much more difficult to calculate solutions. There are, however, some cases for which solutions are known and I shall discuss a few, also giving the appropriate reference for further reading.

a) Translational invariance, constant gravitational force

I will also assume that we have an isothermal plasma, i.e. that $T =$ constant. As we have seen in Sect. 3.4.1 we can then write

$$p = p_0(A)\exp(-z/H) \tag{3.241}$$

$$B_y = B_y(A) \tag{3.242}$$

$$\mu_0 j_y = \frac{\partial}{\partial A}\left[\mu_0 p_0(A)\exp(-z/H) + \frac{1}{2}B_y^2(a)\right]. \tag{3.243}$$

We do not have to discuss the case $\mu_0 j_y = 0$ again, because the solutions are identical to those discussed before.

If

$$\mu_0 j_y = \mu_0 k_1 \exp(-z/H) + k_2 \tag{3.244}$$

(corresponding to the case of constant current density without gravitation) one can find solutions by adding a particular solution of the full equation to any potential solution.

If the current density is linear in A it has the general form

$$\mu_0 j_y = (\mu_0 k_1 \exp(-z/H) + k_2)\, A. \tag{3.245}$$

This case has been discussed for $k_2 = 0$ by Zweibel and Hundhausen (1982).

An interesting case for solar physics is given by If the current density is linear in A it has the general form

$$\mu_0 j_y = \lambda \exp(2A)\exp(-z/H) + B_y \frac{dB_y}{dA} \tag{3.246}$$

If $B_y =$ constant, the second term vanishes. Defining

$$2\bar{A} = 2A - \frac{z}{H} \tag{3.247}$$

we can rewrite the Grad-Shafranov equation in the form

$$-\Delta\bar{A} = \lambda \exp(2\bar{A}). \tag{3.248}$$

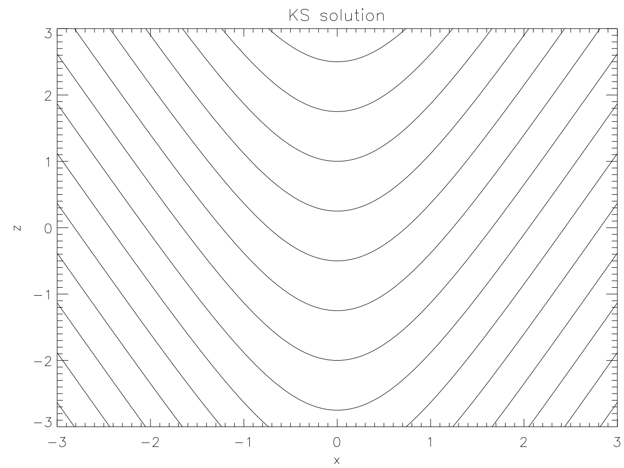

Figure 3.8: Field lines of the Kippenhahn-Schlüter solution.

This equation is Liouville's differential equation again for $\bar{A}$. A particularly well-known solution in solar physics is given by

$$\bar{A} = -\ln\cosh(\sqrt{\lambda}x) \qquad \text{(Harris sheet !).} \tag{3.249}$$

Using the transformation to derive A, we get

$$A = -\ln\cosh(\sqrt{\lambda}x) + \frac{z}{2H} \tag{3.250}$$

and B_y constant. This solution has been discussed by Kippenhahn and Schlüter (1957) as a model for the magnetic support of solar prominences. A field line plot is shown in Fig. 3.8. Further solutions are discussed in Low et al. (1983).

b) Rotational invariance, spherical geometry

Again I assume that T is constant. In this case one gets

$$p = p_0(A)\exp\left(-\frac{GM\mu}{k_BT}\frac{1}{r}\right) \tag{3.251}$$

with μ being the average molecular weight of the plasma. Only very few solutions are known for this problem. Solutions with

$$p_0 = \bar{p}_1 A + \bar{p}_0 \tag{3.252}$$

have been found by Hundhausen et al. (1981) and applied to the solar corona.

For rotating plasmas or if magnetic shear is included, no explicit solutions are known (to the author at least).

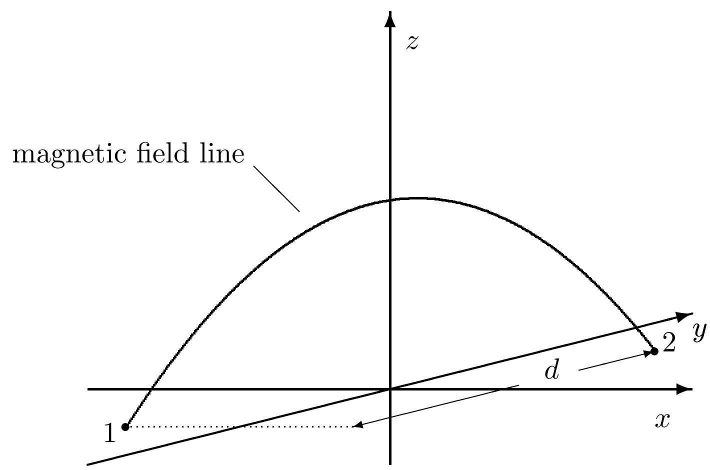

Figure 3.9: Foot point displacement d of a magnetic field line

3.6 Some Remarks about the Choices for p and B_y

So far we have, somewhat naively, assumed that we can prescribe the free functions in the Grad-Shafranov equation as we like. This is, however, usually not the case. To demonstrate this I will consider a problem well-known in solar physics.

Suppose p is constant so that we are dealing with force-free fields. We consider the magnetic field of a coronal arcade which means that the foot points of the magnetic field lines are both anchored in the lower boundary (photosphere). Because B_y will in general not vanish the two foot points will be displaced in the y-direction and we define the *foot point displacement* d of a field line as

$$d = y_2 - y_1, \tag{3.253}$$

where y_1 and y_2 are the y coordinates of the two foot points in the $z = 0$-plane.

Since the plasma in the photosphere is much denser than the plasma in the corona and because it is sufficiently ideal, the motions of the photospheric plasma *determine* the positions of the foot points. In other words: from a physical point of view we should prescribe d for all field lines and not B_y ! Of course, B_y and d are related to each other, the question is how ?

The differential equations for any point (x, y, z) on a field line are

$$\frac{dx}{d\sigma} = \frac{B_x}{|\mathbf{B}|} \tag{3.254}$$

$$\frac{dy}{d\sigma} = \frac{B_y}{|\mathbf{B}|} \tag{3.255}$$

$$\frac{dz}{d\sigma} = \frac{B_z}{|\mathbf{B}|} \tag{3.256}$$

with the arc-length σ along the field line defined by

$$\mathbf{B} \cdot \nabla\sigma = |\mathbf{B}|. \tag{3.257}$$

As $B_y = B_y(A)$ is constant along field lines, we can integrate Eq. (3.255) from foot point 1 to foot point 2 and get

$$d = y_2 - y_1 = B_y(A) \int_{\sigma_1}^{\sigma_2} \frac{1}{|\mathbf{B}|} d\sigma. \tag{3.258}$$

This equation provides a relation between B_y and d, but we can make the meaning of the integral more obvious by introducing a new arc-length s which runs along the *projection* of $\mathbf{B}$ onto the xz-plane, i.e. along contours of A. This arc-length s is defined by

$$\nabla A \times \mathbf{e}_y \cdot \nabla s = \mathbf{B}_p \cdot \nabla s = |\mathbf{B}_p| = |\nabla A| \tag{3.259}$$

The transformation between σ and s along a field line with $A = A_0$ is determined by

$$\frac{ds}{d\sigma} = \sqrt{\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2} = \frac{\sqrt{B_x^2 + B_z^2}}{|\mathbf{B}|} = \left.\frac{|\nabla A|}{|\mathbf{B}|}\right|_{A=A_0}. \tag{3.260}$$

Using this in the expression for d we obtain

$$d(A_0) = B_y(A_0) \int_{s_1}^{s_2} \left.\frac{1}{|\nabla A|}\right|_{A=A_0} ds. \tag{3.261}$$

The integral

$$D(A_0) = \int_{s_1}^{s_2} \left.\frac{1}{|\nabla A|}\right|_{A=A_0} ds \tag{3.262}$$

is usually called the *differential flux volume* (defined per unit length here). This name can be explained as follows. The volume between two flux surfaces defined by $A = A_1$ and $A = A_2$ and the boundary, extending a distance L_y in the invariant direction (see Fig. 3.10) is given by

$$V = \int_{\Omega} dxdydz = L_y \int_{\Omega_2} dxdz = L_y \int_{A_1}^{A_2} \int_{s_1(A)}^{s_2(A)} |J| ds dA \tag{3.263}$$

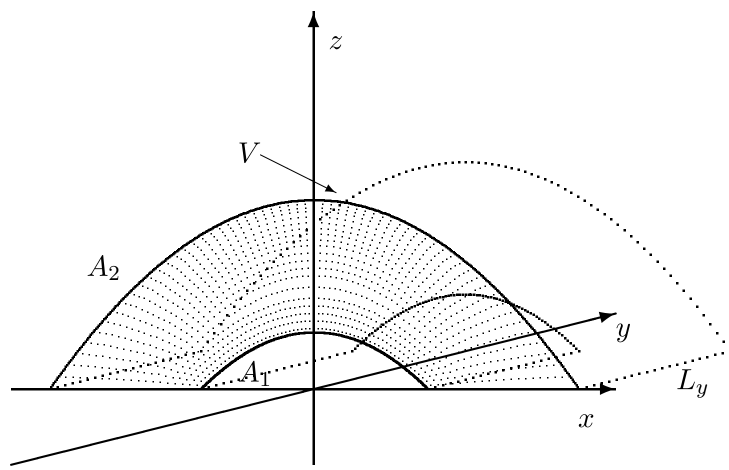

Figure 3.10: Sketch illustrating the geometry of the flux volume V.

The Jacobian J of the transformation from x, z to A, s is reciprocal to the Jacobian of the inverse transformation A, s to x, z which is easier to calculate

$$\begin{vmatrix} \frac{\partial A}{\partial x} & \frac{\partial A}{\partial z} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial z} \end{vmatrix} = \frac{\partial A}{\partial x}\frac{\partial s}{\partial z} - \frac{\partial A}{\partial z}\frac{\partial s}{\partial x} = -\mathbf{B}_p \cdot \nabla s = -|\mathbf{B}_p| = -|\nabla A|. \quad (3.264)$$

We obtain the modulus of the Jacobian determinant as

$$|J| = \frac{1}{|\nabla A|}, \quad (3.265)$$

giving

$$V = L_y \int_{A_1}^{A_2} \int_{s_1(A)}^{s_2(A)} \frac{1}{|\nabla A|} ds dA. \quad (3.266)$$

We now regard the upper limit of the A-integration as a variable and take the derivative of V with respect to this upper limit resulting in

$$\frac{dV}{dA} = L_y \int_{s_1(A)}^{s_2(A)} \frac{1}{|\nabla A|} ds = L_y D(A) \; ! \quad (3.267)$$

This equation explains why $D(A)$ is called the differential flux volume.

So if we prescribe the foot point displacement $d(x)$ on the boundary $z = 0$, we basically prescribe B_y, but we have to solve a nasty integral equation ! We will encounter a simpler way of imposing this sort of boundary condition when we discuss Euler potentials but only at the expense of a far more complicated equilibrium equation itself !

Similar problems occur in other areas of plasma physics as well, e.g. in magnetospheric physics where one has to determine the pressure function in some problems in a similar way as B_y in the case above or in fusion theory, if one wants to prescribe the safety factor $q(A)$ instead of $b_\phi(A)$.

Chapter 4

Non-Symmetric Systems

For non-symmetric systems a general equilibrium theory such as the Grad-Shafranov equation does not exist. Even the very existence of non-symmetric equilibria especially in toroidal geometry has been questioned (Grad, 1985) and is still a matter of ongoing research. Parker (1979) has proved a "non-existence theorem" for non-symmetric equilibria which are calculated by adding "small" perturbations to symmetric equilibria. If generally valid, this would of course limit the usefulness of symmetric equilibria as approximations to non-symmetric equilibria considerably. Fortunately, in his proof Parker uses the assumption that the equilibria exist in the complete R^3 *and* are finite everywhere. As soon as the system has a single boundary (like e.g. the solar surface) the theorem no longer applies !

So in 3D, we do not just face the problem of calculating equilibria, we already have to ask ourselves whether they exist *at all* from a mathematical point of view in specific situations.

4.1 Potential and Linear Force-Free Solutions

4.1.1 Potential (Current Free) Fields

As in the case of symmetric systems we get from the condition

$$\mathbf{j} = \mathbf{0} \tag{4.1}$$

that

$$\nabla \times \mathbf{B} = \mathbf{0} \tag{4.2}$$

and

$$\mathbf{B} = \nabla \Psi. \tag{4.3}$$

It follows that

$$\nabla \cdot \mathbf{B} = \Delta \Psi = 0. \tag{4.4}$$

In the force balance equation we have either

$$\nabla p = \mathbf{0} \qquad \text{(without external forces)} \tag{4.5}$$

implying $p =$ constant or

$$\nabla p = -\rho \nabla \psi \qquad \text{(with external forces)} \tag{4.6}$$

with the plasma in hydrostatic equilibrium.

The basic equation to solve is Laplace's equation

$$\Delta \Psi = 0. \tag{4.7}$$

In three dimensions this can be done by separation of variables in various coordinate systems or by using the method of Green's functions.

In spherical coordinates for example the general solution is given by

$$\Psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(a_{lm} r^l + b_{lm} r^{-(l+1)} \right) \mathrm{Y}_l^m(\theta, \phi) \tag{4.8}$$

where Y_l^m are spherical harmonic functions defined by

$$\mathrm{Y}_l^m(\theta, \phi) = \mathrm{P}_l^m(\cos\theta) \exp(\mathrm{i} m \phi). \tag{4.9}$$

For more information about the solutions of Laplace's equation in three dimensions by separation of variables or Green's functions see e.g. Morse and Feshbach (1953a,b) or Jackson (1975).

4.1.2 Constant Current Fields

Constant current fields are not really interesting in 3D, because of the following arguments. Without loss of generality we can choose

$$\mathbf{j} = j_0 \mathbf{e}_z. \tag{4.10}$$

From

$$\mathbf{j} \cdot \nabla p = 0 \tag{4.11}$$

we get

$$\frac{\partial p}{\partial z} = 0, \tag{4.12}$$

i.e. p depends only on two coordinates. From

$$\nabla \times \mathbf{B} = \mu_0 j_0 \mathbf{e}_z \tag{4.13}$$

we can conclude that e.g.

$$\mathbf{B} = \nabla \Phi + \mu_0 j_0 x \mathbf{e}_y. \tag{4.14}$$

Substitution into the force balance equation results in

$$\begin{aligned}
\mathbf{j} \times \mathbf{B} &= j_0 \mathbf{e}_z \times (\nabla\Phi + \mu_0 j_0 x \mathbf{e}_y) \\
&= \mathbf{e}_z \times \nabla(j_0\Phi) - \mu_0 j_0^2 x \mathbf{e}_x \\
&= \mathbf{e}_z \times \nabla(j_0\Phi) - \nabla\left(\frac{1}{2}\mu_0 j_0^2 x^2\right) = \nabla p.
\end{aligned} \tag{4.15}$$

It follows that

$$-\nabla \times (j_0 \Phi \mathbf{e}_z) = \nabla\left(p + \frac{1}{2}\mu_0 j_0^2 x^2\right). \tag{4.16}$$

Taking the curl of this equation gives

$$\begin{aligned}
\nabla \times (\nabla \times (j_0 \Phi \mathbf{e}_z)) &= 0 \qquad \Longrightarrow \\
\nabla\left(\nabla \cdot (\Phi \mathbf{e}_z)\right) - \mathbf{e}_z \Delta\Phi &= 0.
\end{aligned} \tag{4.17}$$

Writing out the components of Eq. (4.17) we get

$$\frac{\partial^2 \Phi}{\partial x \partial z} = 0 \tag{4.18}$$

$$\frac{\partial^2 \Phi}{\partial y \partial z} = 0 \tag{4.19}$$

$$\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = 0. \tag{4.20}$$

The general solution to this set of equations is

$$\Phi(x, y, z) = Q(x, y) + R(z) \tag{4.21}$$

with

$$\Delta Q = 0. \tag{4.22}$$

On the other hand, Φ has to satisfy

$$\Delta\Phi = \nabla \cdot \mathbf{B} = 0 \tag{4.23}$$

which is equivalent to

$$\frac{d^2 R}{dz^2} = 0 \tag{4.24}$$

implying that

$$R = R_1 z + R_0. \tag{4.25}$$

The only component of $\mathbf{B}$ which can depend on z is B_z, but now we get

$$B_z = \frac{\partial \Phi}{\partial z} = R_1. \tag{4.26}$$

So B_z is a constant and we conclude that the physical quantities depend only on x and y. Therefore we have a translational symmetry in the z-direction.

4.1.3 Linear Force Free Fields

For non-symmetric systems we have the same condition for force-free fields as for symmetric systems, namely

$$\mathbf{j} \times \mathbf{B} = \mathbf{0}. \tag{4.27}$$

It follows that

$$\mu_0 \mathbf{j} = \alpha \mathbf{B}. \tag{4.28}$$

Since

$$\nabla \cdot \mathbf{j} = 0 \tag{4.29}$$

α has to satisfy the condition

$$\mathbf{B} \cdot \nabla \alpha = 0. \tag{4.30}$$

This condition is automatically satisfied if α is constant. One then has to solve

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \tag{4.31}$$

which is a linear equation for $\mathbf{B}$ if α is constant.

There are several different ways to tackle the solution of this equation. For example, if we take the curl of Ampère's law, we get

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\underbrace{\nabla \cdot \mathbf{B}}_{=0}) - \Delta \mathbf{B} = \alpha \nabla \times \mathbf{B} = \alpha^2 \mathbf{B}, \tag{4.32}$$

so we have to solve the vector Helmholtz equation

$$\Delta \mathbf{B} + \alpha^2 \mathbf{B} = 0. \tag{4.33}$$

We could now solve this equation directly either by separation of variables or by Green's function techniques. The problem is to find a solution which satisfies the condition $\nabla \cdot \mathbf{B} = 0$. We will use a different approach which guarantees $\nabla \cdot \mathbf{B} = 0$ from the outset.

Since $\nabla \cdot \mathbf{B} = 0$ we can represent $\mathbf{B}$ by a vector potential $\mathbf{A}$

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{4.34}$$

but again this is not very useful unless we gauge $\mathbf{A}$ in a convenient way, e.g. using the *Coulomb gauge*

$$\nabla \cdot \mathbf{A} = 0 \tag{4.35}$$

Note that this still does not fix $\mathbf{A}$ completely, because we could define $\mathbf{A}' = \mathbf{A} + \nabla \Phi$ with $\Delta \Phi = 0$ satisfying the same gauge condition and giving the same magnetic field $\mathbf{B}$.

Although $\mathbf{A}$ is a three component vector field, it is clear that the gauge condition $\nabla \cdot \mathbf{A} = 0$ leaves only two components independent. This is true whatever gauge we choose, so in principle it must be possible to express $\mathbf{B}$ by two scalar functions T and P representing the two degrees of freedom of $\mathbf{A}$ (or respectively $\mathbf{B}$).

Suppose we choose as one of these two functions the component of $\mathbf{A}$ along a constant unit vector $\mathbf{c}$, i.e.

$$T = \mathbf{c} \cdot \mathbf{A}. \tag{4.36}$$

The other two components of $\mathbf{A}$ should be perpendicular to $\mathbf{c}$ and derivable from a single scalar function P. we can fulfill these two conditions by

$$\mathbf{A} - (\mathbf{c} \cdot \mathbf{A})\mathbf{c} = \nabla P \times \mathbf{c} = \nabla \times (P\mathbf{c}) \tag{4.37}$$

so that

$$\mathbf{A} = \nabla \times (P\mathbf{c}) + T\mathbf{c}. \tag{4.38}$$

Note that this does not in general satisfy the Coulomb gauge condition. In Cartesian coordinates this form of $\mathbf{A}$ can be used for arbitrary $\mathbf{c}$ and in that case we get

$$\mathbf{B} = \nabla \times \nabla \times (P\mathbf{c}) + \nabla \times (T\mathbf{c}). \tag{4.39}$$

The use of the letters P and T is motivated by the name *tangential* component, because the second term is tangential to any plane perpendicular to $\mathbf{c}$, and *parallel* component because the first term is parallel to $\mathbf{c}$.

But P and T are not unique. We get the same magnetic field $\mathbf{B}$ from another set of functions $P' = P + \Phi$ and $T' = T + \Psi$, if

$$\begin{aligned} \nabla \times \nabla \times (\Phi\mathbf{c}) + \nabla \times (\Psi\mathbf{c}) &= 0 \qquad \Longrightarrow \\ \nabla(\nabla \cdot (\Phi\mathbf{c})) - \mathbf{c}\Delta\Phi + \nabla\Psi \times \mathbf{c} &= 0 \end{aligned} \tag{4.40}$$

Let us now take $\mathbf{c} = \mathbf{e}_z$ without loss of generality. Then the z-component of the gauge condition (4.40) is

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{4.41}$$

with the solution

$$\Phi = \Re(f(x + \mathrm{i}y, z)) \tag{4.42}$$

with $f(u, z)$ being an analytic function of $u = x + \mathrm{i}y$. The other two components of the gauge condition are

$$\nabla_2\Psi \times \mathbf{e}_z + \nabla_2\frac{\partial\Phi}{\partial z} = 0 \tag{4.43}$$

where $\nabla_2 = \mathbf{e}_x\partial/\partial x + \mathbf{e}_y\partial/\partial y$. Now $\partial\Phi/\partial z$ is also an analytic function and the two components of the gauge condition are

$$-\frac{\partial\Psi}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial z}\right) \tag{4.44}$$

$$\frac{\partial\Psi}{\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial\Phi}{\partial z}\right). \tag{4.45}$$

These are the Cauchy-Riemann equations for Ψ and $\partial\Phi/\partial z$ and we conclude that

$$\Psi = -\Im\left(\frac{\partial f}{\partial z}\right). \tag{4.46}$$

So Φ and Ψ are determined up to an arbitrary analytic function $f(x+\mathrm{i}y, z)$.

We will now use this result to derive the equations that P and T have to satisfy in the case of linear force-free fields. Because

$$\begin{aligned}\nabla\times\mathbf{B} &= \nabla\times[\nabla(\nabla\cdot(P\mathbf{e}_z)) - \mathbf{e}_z\Delta P] + \nabla\times\nabla\times(T\mathbf{e}_z)\\ &= \nabla\times(-\mathbf{e}_z\Delta P) + \nabla\times\nabla\times(T\mathbf{e}_z)\\ &= \nabla\times(\nabla\times(\alpha P\mathbf{e}_z)) + \nabla\times(\alpha T\mathbf{e}_z) = \alpha\mathbf{B}\end{aligned} \tag{4.47}$$

we get

$$-\nabla\times[(\alpha T + \Delta P)\mathbf{e}_z] + \nabla\times\{\nabla\times[(T-\alpha P)\mathbf{e}_z]\} = 0. \tag{4.48}$$

This is the same equation as for the gauge functions Φ and Ψ so that we can set

$$-\alpha T - \Delta P = \Im\left(\frac{\partial f}{\partial z}\right) \tag{4.49}$$

$$T - \alpha P = \Re(f). \tag{4.50}$$

The most convenient gauge is obviously $f = 0$ with the result

$$T = \alpha P \tag{4.51}$$

$$\Delta P - \alpha^2 P = 0. \tag{4.52}$$

We have replaced the vector Helmholtz equation by a scalar Helmholtz equation and we do not have to pay attention to the solenoidal condition for $\mathbf{B}$.

The basic task is now to solve Eq. (4.52) under suitable boundary conditions. We will usually rather impose these boundary conditions on $\mathbf{B}$ than an P. A typical example in solar physics is that we solve for $\mathbf{B}$ (P) in the half space $z \geq 0$ with $B_z(x, y, 0)$ given. If we do not want to enclose the system in a finite box, we would typically impose the condition that $\mathbf{B}$ goes to zero at infinity. We can then write P as a Fourier integral in x and y because it is bounded as $x^2 + y^2$ goes to infinity

$$P(x, y, z) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \tilde{P}(k_x.k_y, z)\mathrm{e}^{\mathrm{i}(k_x x + k_y y)}. \tag{4.53}$$

Substituting this expression into Eq. (4.52) we see that $\tilde{P}$ obeys the equation

$$\frac{d^2\tilde{P}}{dz^2} + (\alpha^2 - k^2)\tilde{P} = 0 \tag{4.54}$$

with $k^2 = k_x^2 + k_y^2$. The solution to this equation is given by

$$\begin{array}{lcl} k^2 < \alpha^2 & : & \tilde{P} = \tilde{P}_1 \sin(\sqrt{\alpha^2 - k^2} z) + \tilde{P}_2 \cos(\sqrt{\alpha^2 - k^2} z) \\ k^2 > \alpha^2 & : & \tilde{P} = \tilde{P}_3 \exp(\sqrt{k^2 - \alpha^2} z) + \tilde{P}_4 \exp(-\sqrt{k^2 - \alpha^2} z) \end{array} . \tag{4.55}$$

By using the condition that P should vanish for $z \to \infty$, we can only discard the exponentially growing solution, but we cannot discard any of the solutions for $k^2 < \alpha^2$ because the character of these solutions for $z \to 0$ will only become obvious *after* the k_x, k_y integration has been carried out.

I would like to remark at this point that although in the solar physics literature you will find mainly solutions which are periodic in x and y one can represent *any* non-periodic function with the suitable properties by a Fourier integral.

Using the expressions for $\tilde{P}$ we can write the general solution for P as

$$\begin{aligned} P(x,y,z) \quad = \quad & \iint\limits_{k^2>\alpha^2} dk_x dk_y \tilde{P}_4(k_x,k_y) \exp(-\sqrt{k^2-\alpha^2} z) \exp[\mathrm{i}(k_x x + k_y y)] \\ & + \iint\limits_{k^2 \le \alpha^2} dk_x dk_y \Big[\tilde{P}_1(k_x,k_y) \sin(\sqrt{\alpha^2 - k^2} z) \\ & \qquad + \tilde{P}_2 \cos(\sqrt{\alpha^2 - k^2} z) \Big] \exp[\mathrm{i}(k_x x + k_y y)]. \end{aligned} \tag{4.56}$$

This expression is not very compact and it can be written in a more useful form in cylindrical coordinates ϖ, ϕ, z with $\varpi^2 = x^2 + y^2$, $x = \varpi \cos\phi$, $y = \varpi \sin\phi$. We can either solve the original equation in cylindrical coordinates or transform the integrals directly. Both ways are instructive, but we choose the second possibility.

We start by introducing polar coordinates in the k_x, k_y plane by defining

$$k_x = k \sin\theta \tag{4.57}$$

$$k_y = k \cos\theta. \tag{4.58}$$

With this definition we get

$$\begin{aligned} \exp[\mathrm{i}(k_x x + k_y y)] &= \exp[\mathrm{i} k \varpi (\sin\theta \cos\phi + \cos\theta \sin\phi)] \\ &= \exp[\mathrm{i} k \varpi \sin(\theta + \phi)]. \end{aligned} \tag{4.59}$$

Now

$$\exp(\mathrm{i}\xi \sin\alpha) = \sum_{n=-\infty}^{\infty} \exp(\mathrm{i}n\alpha) \mathrm{J}_n(\xi), \tag{4.60}$$

with $\mathrm{J}_n(x)$ a Bessel function, so we obtain

$$\exp(\mathrm{i}k_x x + \mathrm{i}k_y y) = \sum_{n=-\infty}^{\infty} \exp[\mathrm{i}n(\theta + \phi)] \mathrm{J}_n(k\varpi). \tag{4.61}$$

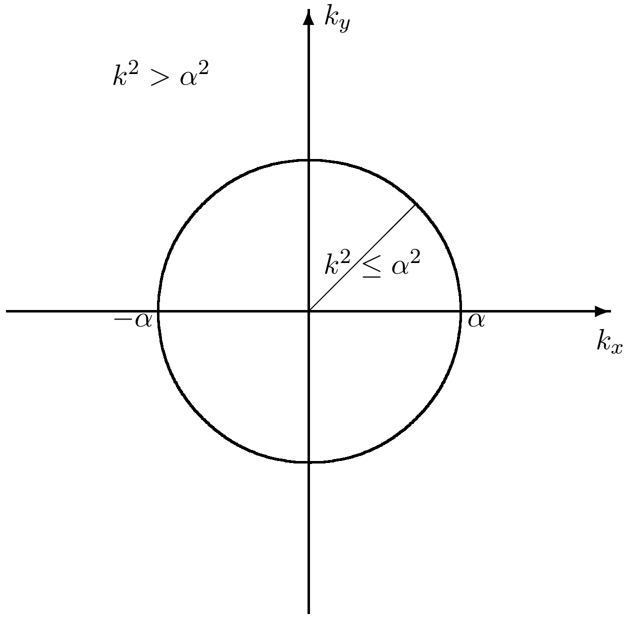

Figure 4.1: Integration domain in the k_x, k_y plane.

When changing the integration variables from k_x, k_y to k, θ, we have to pay attention to the domain of integration as sketched in Fig. 4.1. If we do that, we obtain

$$
\begin{aligned}
&\iint\limits_{k^2>\alpha^2} dk_x dk_y \tilde{P}_4(k_x, k_y) \exp(-\sqrt{k^2-\alpha^2}z) \exp[\mathrm{i}(k_x x + k_y y)] \\
&= \int\limits_{\alpha}^{\infty} dk\, k \int\limits_{0}^{2\pi} d\theta \tilde{P}_4(k\sin\theta, k\cos\theta) \exp(-\sqrt{k^2-\alpha^2}z) \\
&\qquad \sum_{n=-\infty}^{\infty} \exp[\mathrm{i}n(\theta+\phi)] \mathrm{J}_n(k\varpi) \\
&= \sum_{n=-\infty}^{\infty} \exp(\mathrm{i}n\phi) \int\limits_{\alpha}^{\infty} dk\, k \exp(-\sqrt{k^2-\alpha^2}z) \mathrm{J}_n(k\varpi) \\
&\qquad \underbrace{\int\limits_{0}^{2\pi} d\theta \tilde{P}_4(k\sin\theta, k\cos\theta) \exp(\mathrm{i}n\theta)}_{=A_n(k)/k} \\
&= \sum_{n=-\infty}^{\infty} \exp(\mathrm{i}n\phi) \int\limits_{\alpha}^{\infty} dk\, A_n(k) \exp(-\sqrt{k^2-\alpha^2}z) \mathrm{J}_n(k\varpi). \qquad (4.62)
\end{aligned}
$$

In a similar way we get for the second integral

$$\int_{-\infty}^{\infty} dk_x \int_{-\sqrt{\alpha^2-k^2}}^{\sqrt{\alpha^2-k^2}} dk_y \left[\tilde{P}_1(k_x,k_y)\sin(\sqrt{\alpha^2-k^2}z)+\right.$$
$$\left.\tilde{P}_2\cos(\sqrt{\alpha^2-k^2}z)\right]\exp[\mathrm{i}(k_x x+k_y y)]$$
$$= \sum_{n=-\infty}^{\infty}\exp(\mathrm{i}n\phi)\int_0^{\alpha} dk\left[B_n(k)\cos(\sqrt{\alpha^2-k^2}z)+\right.$$
$$\left.C_n(k)\sin(\sqrt{\alpha^2-k^2}z)\right]\mathrm{J}_n(k\varpi). \tag{4.63}$$

This is the same expression as given in Chiu and Hilton (1977) (note that their expression (5) contains a typo; the lower limit of the first integral should be α, not 0).

We now want to prescribe $B_z(x,y,0) = B_z(\varpi,\phi,0)$ and calculate the functions $A_n(k)$, $B_n(k)$ and $C_n(k)$ from this boundary condition. It turns out that only A_n and B_n are fixed in this way. One needs additional information to fix C_n. By evaluating B_z from

$$\mathbf{B} = \nabla\times[\nabla\times(P\mathbf{e}_z)]+\alpha\nabla\times(P\mathbf{e}_z) \tag{4.64}$$

we get

$$B_z = -\frac{\partial^2 P}{\partial x^2}-\frac{\partial^2 P}{\partial y^2} = \frac{\partial^2 P}{\partial z^2}+\alpha^2 P \tag{4.65}$$

where we have used Eq. (4.52) in the second equality. It follows that

$$B_z(r,\phi,0) = \sum_{n=-\infty}^{\infty}\exp(\mathrm{i}n\phi)\Bigg\{$$
$$\int_{\alpha}^{\infty} dk\, A_n(k)(k^2-\alpha^2)\overbrace{\exp(-\sqrt{k^2-\alpha^2}0)}^{=1}\,\mathrm{J}_n(k\varpi)$$
$$+\int_0^{\alpha} dk B_n(k)[-(\alpha^2-k^2)\overbrace{\cos(\sqrt{\alpha^2-k^2}0)}^{=1}]\mathrm{J}_n(k\varpi)$$
$$+\int_0^{\alpha} dk C_n(k)[-(\alpha^2-k^2)\overbrace{\sin(\sqrt{\alpha^2-k^2}0)}^{=0}]\mathrm{J}_n(k\varpi)$$
$$+\int_{\alpha}^{\infty} dk\, A_n(k)\alpha^2)\overbrace{\exp(-\sqrt{k^2-\alpha^2}0)}^{=1}\,\mathrm{J}_n(k\varpi)$$
$$+\int_0^{\alpha} dk B_n(k)\alpha^2\overbrace{\cos(\sqrt{\alpha^2-k^2}0)}^{=1}\,\mathrm{J}_n(k\varpi)$$

$$+\int_0^\alpha dk C_n(k)\alpha^2 \overbrace{\sin(\sqrt{\alpha^2-k^2}0)}^{=0} \mathrm{J}_n(k\varpi)\Bigg\} \tag{4.66}$$

leading finally to

$$B_z(r,\phi,0) = \sum_{n=-\infty}^{\infty} \exp(\mathrm{i}n\phi)\Bigg\{\int_\alpha^\infty dk\, k^2 A_n(k)\mathrm{J}_n(k\varpi) + \int_0^\alpha dk k^2 B_n(k)\mathrm{J}_n(k\varpi)\Bigg\}. \tag{4.67}$$

So $C_n(k)$ cannot be determined by this boundary condition ! We now have to invert the equation above to determine $A_n(k)$ and $B_n(k)$ from $B_z(\varpi,\phi,0)$. We will need a few useful equations to be able to carry this out. In particular we will need the completeness relations for the Bessel functions

$$\int_0^\infty dx\, x\mathrm{J}_m(\lambda x)\mathrm{J}_m(\lambda' x) = \frac{1}{\lambda'}\delta(\lambda'-\lambda) \tag{4.68}$$

and for $\exp(\mathrm{i}n\phi)$

$$\int_0^{2\pi} d\phi \exp(\mathrm{i}n\phi)\exp(-\mathrm{i}m\phi) = 2\pi\delta_{nm}. \tag{4.69}$$

The strategy is the following.

a) Multiply Eq. (4.67) by $\mathrm{J}_m(k')\exp(-\mathrm{i}m\phi)$.

b) Integrate over $\int_0^\infty d\varpi\,\varpi\int_0^{2\pi}$.

With this strategy we obtain

$$\begin{aligned}\int_0^\infty d\varpi\,\varpi\int_0^{2\pi} d\phi\ \exp(-\mathrm{i}m\phi)\mathrm{J}_m(k'\varpi)B_z(\varpi,\phi,0) \\ = \int_0^\infty d\varpi\,\varpi\int_0^{2\pi} d\phi\ \left(\sum_{n=-\infty}^{\infty}\exp(\mathrm{i}n\phi)\exp(-\mathrm{i}m\phi)\right) \\ \Bigg[\int_\alpha^\infty dk\, k^2 A_n(k)\mathrm{J}_n(k\varpi)\mathrm{J}_m(k'\varpi) \\ + \int_0^\alpha dk k^2 B_n(k)\mathrm{J}_n(k\varpi)\mathrm{J}_m(k'\varpi)\Bigg]\end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{\int_0^{2\pi} \exp(\mathrm{i}n\phi)\exp(-\mathrm{i}m\phi)\,d\phi}_{=2\pi\delta_{nm}}$$

$$\left\{ \int_\alpha^\infty dk\,k^2 A_n(k) \left[\underbrace{\int_0^\infty d\varpi\,\varpi \mathrm{J}_n(k\varpi)\mathrm{J}_m(k'\varpi)}_{=\delta(k'-k)/k'} \right] + \right.$$

$$\left. \int_0^\alpha dk k^2 B_n(k) \left[\underbrace{\int_0^\infty d\varpi\,\varpi \mathrm{J}_n(k\varpi)\mathrm{J}_m(k'\varpi)}_{=\delta(k'-k)/k'} \right] \right\}$$

$$= 2\pi \left[k' A_m(k')\Theta(k'-\alpha) + k' B_m(k')\Theta(\alpha - k') \right]. \tag{4.70}$$

Renaming k' as k again, we get

$$2\pi \left[k A_m(k)\Theta(k-\alpha) + k B_m(k)\Theta(\alpha-k) \right] = \int_0^\infty d\varpi\,\varpi \int_0^{2\pi} d\phi\ \exp(-\mathrm{i}m\phi)\mathrm{J}_m(k'\varpi)B_z(\varpi,\phi,0) \tag{4.71}$$

or

$$\begin{array}{lcl} k > \alpha & : & A_m(k) = \frac{1}{2\pi k}\int_0^\infty d\varpi\,\varpi \int_0^{2\pi} d\phi \exp(-\mathrm{i}m\phi)\mathrm{J}_m(k\varpi)B_z(\varpi,\phi,0) \\ k \le \alpha & : & B_m(k) = \frac{1}{2\pi k}\int_0^\infty d\varpi\,\varpi \int_0^{2\pi} d\phi \exp(-\mathrm{i}m\phi)\mathrm{J}_m(k\varpi)B_z(\varpi,\phi,0) \end{array}. \tag{4.72}$$

Since α corresponds to an inverse length scale L_α, say,

- A_m represents the small scale features of the boundary data B_z: $k > \alpha \Longrightarrow L_k < L_\alpha$.
- B_m represents the large scale features of the boundary data B_z: $k \le \alpha \Longrightarrow L_k \ge L_\alpha$.

We can now substitute these expressions for A_m and B_m back into the first two terms of our expression for P and get

$$\sum_{n=-\infty}^{\infty} \exp(\mathrm{i}n\phi) \left[\int_\alpha^\infty dk A_n(k) \exp(-\sqrt{k^2-\alpha^2}z)\mathrm{J}_n(k\varpi) + \int_0^\alpha dk B_n(k)\cos(\sqrt{\alpha^2-k^2}z)\mathrm{J}_n(k\varpi) \right]$$

$$= \sum_{n=-\infty}^{\infty} \exp(in\phi)$$

$$\Bigg\{ \int_\alpha^\infty dk \left[\frac{1}{2\pi k} \int_0^\infty d\varpi' \, \varpi' \int_0^{2\pi} d\phi' \exp(-in\phi') \mathrm{J}_n(k\varpi') B_z(\varpi', \phi', 0) \right]$$

$$\exp(-\sqrt{k^2 - \alpha^2} z) \mathrm{J}_n(k\varpi)$$

$$\int_0^\alpha dk \left[\frac{1}{2\pi k} \int_0^\infty d\varpi' \, \varpi' \int_0^{2\pi} d\phi' \exp(-in\phi') \mathrm{J}_n(k\varpi') B_z(\varpi', \phi', 0) \right]$$

$$\cos(\sqrt{\alpha^2 - k^2} z) \mathrm{J}_n(k\varpi) \Bigg\}. \tag{4.73}$$

Now

$$\sum_{n=-\infty}^{\infty} \exp(in\phi) \exp(in\phi') \mathrm{J}_n(k\varpi) \mathrm{J}_n(k\varpi') = \mathrm{J}_0(kR) \tag{4.74}$$

with

$$R^2 = (x - x')^2 + (y - y')^2 = \varpi^2 + \varpi'^2 - 2\varpi\varpi' \cos(\phi - \phi'). \tag{4.75}$$

If we denote the first two terms of P by P_1, we can write

$$P_1(\varpi, \phi, z) = \frac{1}{2\pi} \int_0^\infty d\varpi' \, \varpi' \int_0^{2\pi} d\phi' \; G_1(\varpi, \phi, z; \varpi', \phi', 0) B_z(\varpi', \phi', 0) \tag{4.76}$$

with

$$\begin{aligned} G_1(\varpi, \phi, z; \varpi', \phi', 0) &= \int_\alpha^\infty dk \frac{1}{k} \mathrm{J}_0(kR) \exp(-\sqrt{k^2 - \alpha^2} z) \\ &\quad + \int_0^\alpha dk \frac{1}{k} \mathrm{J}_0(kR) \cos(\sqrt{\alpha^2 - k^2} z). \end{aligned} \tag{4.77}$$

We can derive the contribution to $\mathbf{B}$ resulting from P_1 and therefore from B_z by calculating

$$\begin{aligned} \mathbf{B} &= \nabla \times \nabla (P_1 \mathbf{e}_z) + \alpha \nabla P_1 \times \mathbf{e}_z \\ &= \nabla \frac{\partial P_1}{\partial z} - \mathbf{e}_z \Delta P_1 + \alpha \left(\frac{\partial P_1}{\partial y} \mathbf{e}_x - \frac{\partial P_1}{\partial x} \mathbf{e}_y \right). \end{aligned} \tag{4.78}$$

Thus we obtain

$$B_{x1} = \frac{\partial^2 P_1}{\partial x \partial z} + \alpha \frac{\partial P_1}{\partial y} \tag{4.79}$$

$$B_{y1} = \frac{\partial^2 P_1}{\partial y \partial z} - \alpha \frac{\partial P_1}{\partial x} \tag{4.80}$$

$$B_{z1} = -\frac{\partial^2 P_1}{\partial x^2} - \frac{\partial^2 P_1}{\partial y^2} = \frac{\partial^2 P_1}{\partial z^2} + \alpha^2 P_1. \tag{4.81}$$

To evaluate these formulae we need the derivatives of the Green's function $\mathbf{G}_1$. Using Eq. (4.77) we get

$$\begin{aligned}
\frac{\partial G_1}{\partial x} = \int\limits_{\alpha}^{\infty} dk \frac{1}{k} k \frac{\partial R}{\partial x} \mathrm{J}_0'(kR) \exp(-\sqrt{k^2-\alpha^2} z) \\
+ \int\limits_{0}^{\alpha} dk \frac{1}{k} k \frac{\partial R}{\partial x} \mathrm{J}_0'(kR) \cos(\sqrt{\alpha^2-k^2} z) \\
= \quad -\frac{x-x'}{R} \left[\int\limits_{\alpha}^{\infty} \mathrm{J}_1(kR) \exp(-\sqrt{k^2-\alpha^2} z) + \right. \\
\left. \int\limits_{0}^{\alpha} dk \mathrm{J}_1(kR) \cos(\sqrt{\alpha^2-k^2} z) \right] \qquad (4.82)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial G_1}{\partial y} = -\frac{x-x'}{R} \left[\int\limits_{\alpha}^{\infty} \mathrm{J}_1(kR) \exp(-\sqrt{k^2-\alpha^2} z) + \right. \\
\left. \int\limits_{0}^{\alpha} dk \mathrm{J}_1(kR) \cos(\sqrt{\alpha^2-k^2} z) \right]. \qquad (4.83)
\end{aligned}$$

Introducing

$$\bar{\Gamma} = -\int\limits_{\alpha}^{\infty} dk \mathrm{J}_1(kR) \exp(-\sqrt{k^2-\alpha^2} z) - \int\limits_{0}^{\alpha} dk \mathrm{J}_1(kR) \cos(\sqrt{\alpha^2-k^2} z) \quad (4.84)$$

we get

$$B_{x1} = \frac{1}{2\pi} \int\limits_{0}^{\infty} d\varpi' \varpi' \int d\phi' \left[\frac{x-x'}{R} \frac{\partial \bar{\Gamma}}{\partial z} + \alpha \frac{y-y'}{R} \bar{\Gamma} \right] B_z(\varpi', \phi', 0) \quad (4.85)$$

$$B_{y1} = \frac{1}{2\pi} \int\limits_{0}^{\infty} d\varpi' \varpi' \int d\phi' \left[\frac{y-y'}{R} \frac{\partial \bar{\Gamma}}{\partial z} - \alpha \frac{x-x'}{R} \bar{\Gamma} \right] B_z(\varpi', \phi', 0) \quad (4.86)$$

$$\begin{aligned}
B_{z1} = & -\frac{1}{2\pi} \int\limits_{0}^{\infty} d\varpi' \varpi' \int d\phi' \left[\frac{\partial}{\partial x} \left(\frac{x-x'}{R} \bar{\Gamma} \right) + \right. \\
& \left. \frac{\partial}{\partial y} \left(\frac{y-y'}{R} \bar{\Gamma} \right) \right] B_z(\varpi', \phi', 0) \\
= & -\frac{1}{2\pi} \int\limits_{0}^{\infty} d\varpi' \varpi' \int d\phi' \left[\frac{x-x'}{R} \frac{\partial \bar{\Gamma}}{\partial x} + \frac{y-y'}{R} \frac{\partial \bar{\Gamma}}{\partial y} \right. \\
& \left. + \frac{R-(x-x')^2 \frac{1}{R}}{R^2} \bar{\Gamma} + \frac{R-(y-y')^2 \frac{1}{R}}{R^2} \bar{\Gamma} \right] B_z(\varpi', \phi', 0)
\end{aligned}$$

$$= \quad -\frac{1}{2\pi}\int_0^\infty d\varpi'\varpi'\int d\phi'\left[\frac{x-x'}{R}\frac{\partial\bar{\Gamma}}{\partial x}\right.$$

$$\left. + \frac{y-y'}{R}\frac{\partial\bar{\Gamma}}{\partial y} + \frac{\bar{\Gamma}}{R}\right] B_z(\varpi',\phi',0). \qquad (4.87)$$

If we introduce the angle Θ by defining

$$x - x' \quad = \quad R\cos\Theta \qquad (4.88)$$

$$y - y' \quad = \quad R\sin\Theta, \qquad (4.89)$$

we get

$$\frac{\partial\bar{\Gamma}}{\partial R} = \frac{\partial x}{\partial R}\frac{\partial\bar{\Gamma}}{\partial x} + \frac{\partial y}{\partial R}\frac{\partial\bar{\Gamma}}{\partial y} = \cos\Theta\frac{\partial\bar{\Gamma}}{\partial x} + \sin\Theta\frac{\partial\bar{\Gamma}}{\partial y}. \qquad (4.90)$$

Using this in the expression for B_{z1} we obtain

$$B_{z1} = -\frac{1}{2\pi}\int_0^\infty d\varpi'\varpi'\int d\phi'\left(\frac{\partial\bar{\Gamma}}{\partial R} + \frac{\bar{\Gamma}}{R}\right) B_z(\varpi',\phi',0). \qquad (4.91)$$

The form of $\bar{\Gamma}$ we have derived so far is a little bit inconvenient because it still involves an integration over k. Thererfore, we need to derive a closed form for $\bar{\Gamma}$ in the following way:

$$\bar{\Gamma} = \int_\alpha^\infty dk\, \underbrace{(-\mathrm{J}_1(kR))}_{=\frac{1}{R}\frac{d\mathrm{J}_0}{dk}} \exp(-\sqrt{k^2-\alpha^2}z) +$$

$$\int_0^\alpha dk\, \underbrace{(-\mathrm{J}_1(kR))}_{=\frac{1}{R}\frac{d\mathrm{J}_0}{dk}} \cos(\sqrt{\alpha^2-k^2}z)$$

$$= \frac{1}{R}\left[\exp(-\sqrt{k^2-\alpha^2}z)\mathrm{J}_0(kR)\right]_{k=\alpha}^{\infty}$$

$$+ \frac{z}{R}\int_\alpha^\infty dk\frac{k}{\sqrt{k^2-\alpha^2}}\exp(-\sqrt{k^2-\alpha^2}z)\mathrm{J}_0(kR)$$

$$+ \frac{1}{R}\left[\cos(\sqrt{\alpha^2-k^2}z)\mathrm{J}_0(kR)\right]_{k=0}^{\alpha}$$

$$- \frac{z}{R}\int_0^\alpha dk\frac{k}{\sqrt{\alpha^2-k^2}}\sin(\sqrt{\alpha^2-k^2}z)\mathrm{J}_0(kR)$$

$$= \frac{1}{R}\left[-\mathrm{J}_0(\alpha R) + \mathrm{J}_0(\alpha R) - \cos(\alpha z)\right] +$$

$$\frac{z}{R}\left[\int_{\alpha}^{\infty} dk \frac{k}{\sqrt{k^2-\alpha^2}} \exp(-\sqrt{k^2-\alpha^2}z) \mathrm{J}_0(kR) - \right.$$
$$\left. \int_{0}^{\alpha} dk \frac{k}{\sqrt{\alpha^2-k^2}} \sin(\sqrt{\alpha^2-k^2}z) \mathrm{J}_0(kR) \right]. \qquad (4.92)$$

Now we use the following identity for the integral part citep[can be worked out from][, p. 416, Eq. (4)]watson :

$$\int_{\alpha}^{k} dk \frac{k \exp(-\sqrt{k^2-\alpha^2}z)}{\sqrt{k^2-\alpha^2}} \mathrm{J}_0(kR) - \int_{0}^{\alpha} dk \frac{k \sin(\sqrt{\alpha^2-k^2}z)}{\sqrt{\alpha^2-k^2}} \mathrm{J}_0(kR) = \frac{\cos(\alpha\rho)}{\rho} \qquad (4.93)$$

with $\rho^2 = R^2 + z^2$. So we finally find

$$\bar{\Gamma} = \frac{z}{R\rho} \cos(\alpha\rho) - \frac{\cos(\alpha z)}{R}. \qquad (4.94)$$

With this $\bar{\Gamma}$ we can easily work out what the contribution of G_1 to the different components of $\mathbf{B}$ is. From the integrals we can then work out $\mathbf{B}$ given B_z on the boundary.

We have not yet discussed the contribution of the third term. This turns out to be fairly difficult. It would be most natural to fix C_n by imposing the value of another component of $\mathbf{B}$ on the lower boundary. This is, however, very difficult because C_n cannot be calculated as simply as A_n and B_n from either B_x or B_y.

Chiu and Hilton (1977) suggest to impose the angle

$$\tan\delta(x,y,0) = \frac{B_y(x,y,0)}{B_x(x,y,0)} \qquad (4.95)$$

as additional information. More recently several authors have suggested to minimise the deviation of the calculated field from the measured field on the photosphere to determine C_n (and eventually α !) (Amari et al., 1997; Wheatland, 1999). To my knowledge, nobody has ever calculated equilibria which include the C_n-term so far (Lothian and Browning, 1995, for example neglect the C_n contribution).

What about the third field component ? Once we have fixed two components of $\mathbf{B}$ the condition $\nabla \cdot \mathbf{B} = 0$ fixes the third. If α would not be constant we would have additional freedom but that problem is even more difficult to solve.

But here the problem is even worse because the C_n only represent *large-scale* contributions to $\mathbf{B}$. So we cannot simply just B_x or B_y which might be determined by observations, but only *the large scale part* of them because the small scale contributions is alreadt completely determined by imposing B_z !

This is an example of an *ill-posed* problem. The well-posed problem would be the one with $\mathbf{n} \times \mathbf{B}$ prescribed on the boundary (see e.g. Morse and Feshbach, 1953b, , Chapter 13). The problem is less dramatic if we enclose the system in a finite box and prescribe the normal component of $\mathbf{B}$ on all boundaries.

This is presently the state-of-the art in 3D force-free fields in general. Apart from laminated force-free solutions only very few other 3D force-free solutions are known (Low and Lou, 1990).

4.2 Euler Potentials

For symmetric systems we have been able to formulate the equilibrium problem in terms of a flux function A. The natural question is whether there is a similar formulation for the 3D non-symmetric case. Such a formulation does actually exist and it uses Euler potentials (also called a Clebsch representation) for the magnetic field $\mathbf{B}$. As in the symmetric case we start with an expression for $\mathbf{B}$ which satisfies $\nabla \cdot \mathbf{B} = 0$ automatically.

With the Euler potentials α and β, we can write

$$\mathbf{B} = \nabla\alpha \times \nabla\beta = \nabla \times (\alpha\nabla\beta) \qquad \text{(matched Euler potentials)} \tag{4.96}$$

or

$$\mathbf{B} = f(\alpha', \beta')\nabla\alpha' \times \nabla\beta' \qquad \text{(unmatched Euler potentials)} \tag{4.97}$$

with an arbitrary function $f(x, y)$. Before we investigate how this is related to the flux function of symmetric systems, we first formulate the equilibrium problem in terms of Euler potentials. We will directly deal with the case including external gravitation. The case without external force follows as a special case.

We start with the force balance equation and substitute the Euler potential representation for $\mathbf{B}$

$$\begin{aligned} \mathbf{j} \times \mathbf{B} - \nabla p - \rho\nabla\Psi &= \mathbf{j} \times (\nabla\alpha \times \nabla\beta) - \nabla p - \rho\nabla\Psi \\ &= (\mathbf{j} \cdot \nabla\beta)\nabla\alpha - (\mathbf{j} \cdot \nabla\alpha)\nabla\beta - \nabla p - \rho\nabla\Psi \\ &= \mathbf{0}. \end{aligned} \tag{4.98}$$

As in the symmetric case, $\nabla\alpha$, $\nabla\beta$ and $\nabla\Psi$ represent three linearly independent vector fields and we can split the force balance equation into three components along $\nabla\alpha$, $\nabla\beta$ and $\nabla\Psi$:

$$\mathbf{j} \cdot \nabla\beta - \left(\frac{\partial p}{\partial \alpha}\right)_{\beta,\Psi} = 0 \tag{4.99}$$

$$-\mathbf{j} \cdot \nabla\alpha - \left(\frac{\partial p}{\partial \beta}\right)_{\alpha,\Psi} = 0 \tag{4.100}$$

$$-\left(\frac{\partial p}{\partial \Psi}\right)_{\alpha,\beta} - \rho = 0. \tag{4.101}$$

With

$$\mu_0 \mathbf{j} = \nabla \times \mathbf{B} = \nabla \times (\nabla\alpha \times \nabla\beta) \tag{4.102}$$

we get

$$\nabla\beta \cdot \nabla \times (\nabla\alpha \times \nabla\beta) - \left(\frac{\partial p}{\partial \alpha}\right)_{\beta,\Psi} = 0 \tag{4.103}$$

$$-\nabla\alpha \cdot \nabla \times (\nabla\alpha \times \nabla\beta) - \left(\frac{\partial p}{\partial \beta}\right)_{\alpha,\Psi} = 0 \tag{4.104}$$

$$-\left(\frac{\partial p}{\partial \Psi}\right)_{\alpha,\beta} = \rho. \tag{4.105}$$

As in the symmetric case we will have to supply information about the thermodynamics of the problem to solve the third equation.

The resulting partial differential equations for α and β are a system of non-linear coupled second order equations. Note that even the differential operators are non-linear. Also this system of equations is not of a unique type in terms of elliptic, hyperbolic or parabolic, but of mixed type. From a mathematical point of view it is not clear what a well-posed problem would be.

We have formulated the mathematical problem in terms of Euler potentials, but what are they and what are their properties ? First of all, from their definition we get directly that

$$\mathbf{B} \cdot \nabla\alpha = \mathbf{B} \cdot \nabla\beta = 0, \tag{4.106}$$

so α and β are constant along field lines. This means that we can use α and β to label field lines. A surface $\alpha = \alpha_0 =$ constant is a flux surface consisting of magnetic field lines and the same is true for the surfaces $\beta = \beta_0 =$ constant. The intersection of the two surfaces defines a field line with $\alpha = \alpha_0$ and $\beta = \beta_0$. These flux surfaces are not unique, because any surface consisting of field lines is a flux surface. This is reflected by the fact that Euler potentials are not unique for a given magnetic field $\mathbf{B}$ but can be gauged, i.e. the same magnetic field can be represented by different Euler potentials.

The major disadvantage of Euler potentials, apart from the complexity of the mathematical equations is that they do not always exist *globally* for a given magnetic field $\mathbf{B}$. What does this mean ?

We can always find Euler potentials α and β which represent the magnetic field correctly *locally*, i.e. close to a fixed location $\mathbf{r}_0$. However, in 3D we can only guarantee that the same Euler potentials represent the magnetic field *everywhere*, i.e. also away from $\mathbf{r}_0$ (i.e. *globally*), if a) the domain contains one surface which each field line intersects only once and if b) the magnetic field does not have any null points ($\mathbf{B} = \mathbf{0}$) inside the domain, or if the magnetic field has a vector potential $\mathbf{A}$ for which $\mathbf{A} \cdot \mathbf{B} = 0$ (Rosner et al., 1989).

Condition a) makes Euler potentials more or less useless for non-symmetric equilibria in tori and condition b) can cause difficulties if null points exist inside the considered domain as is often the case in models of coronal magnetic fields. The reason for this is that the magnetic field lines tend to be 'chaotic', i.e. they do usually not define smooth magnetic surfaces. The equations for field lines

$$\frac{d\mathbf{r}}{d\tau} = \frac{\mathbf{B}}{|\mathbf{B}|} \tag{4.107}$$

can be regarded as describing a *dynamical system* and the methods of dynamical system theory can be applied to investigate the behaviour of field lines.

The second possible condition under which global Euler potentials can be shown to exist has the consequence that the total magnetic helicity $H = \int_V \mathbf{A} \cdot \mathbf{B} dV$ of such fields vanishes.

4.2.1 Euler Potentials in Symmetric Systems

Euler potentials are sometimes also useful for symmetric systems. If we recall the expression for the magnetic field in terms of a flux function and compare to the matched Euler potential form in 3D, we see that (in Cartesian coordinates for simplicity)

$$\mathbf{B}^{\text{symm}} = \nabla A \times \mathbf{e}_y + B_y \mathbf{e}_y; \quad A = A(x,z),\ B_y = B_y(x,z) \tag{4.108}$$

$$\mathbf{B}^{\text{Euler}} = \nabla\alpha \times \nabla\beta. \tag{4.109}$$

If we choose $\beta = y + \tilde{\beta}(x,z)$ and assume that $\alpha = \alpha(x,z)$ we get

$$\mathbf{B}^{\text{Euler}} = \nabla\alpha \times \nabla(y + \tilde{\beta}) = \nabla\alpha \times \mathbf{e}_y + \underbrace{\nabla\alpha \times \nabla\tilde{\beta}}_{\parallel\ \mathbf{e}_y}. \tag{4.110}$$

This shows that we can identify α with A and $\nabla\alpha \times \nabla\tilde{\beta}$ with $B_y \mathbf{e}_y$ for symmetric systems. Note that $\tilde{\beta}$ is *not* constant along field lines.

The equilibrium equations for the symmetric Euler potentials reduce to

$$\nabla(\tilde{\beta} + y) \cdot \nabla \times (\nabla\alpha \times \mathbf{e}_y + \nabla\alpha \times \nabla\tilde{\beta}) = \left(\frac{\partial p}{\partial \alpha}\right)_\Psi \tag{4.111}$$

$$-\nabla\alpha \cdot \nabla \times (\nabla\alpha \times \mathbf{e}_y + \nabla\alpha \times \nabla\tilde{\beta}) = 0 \tag{4.112}$$

because p may not depend on y and so it cannot depend on β and thus on $\tilde{\beta}$. We simplify the equilibrium equations and write

$$\nabla(\tilde{\beta} + y) \cdot \left[-\Delta\alpha \mathbf{e}_y + \nabla \times (\overbrace{\nabla\alpha \times \nabla\tilde{\beta}}^{B_y \mathbf{e}_y}) \right] =$$

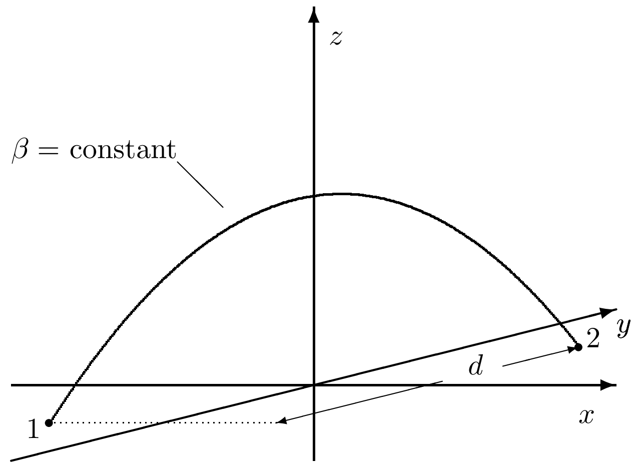

Figure 4.2: Connection between foot point displacement d and Euler potentials.

$$
\begin{aligned}
(\underbrace{\mathbf{e}_y \cdot \nabla\tilde{\beta}}_{=0} + 1)(-\Delta\alpha) + (\nabla\tilde{\beta} + \mathbf{e}_y) \cdot \nabla B_y \times \mathbf{e}_y &= \\
-\Delta\alpha + \nabla\tilde{\beta} \cdot \nabla \times (\nabla\alpha \times \nabla\tilde{\beta}) &= \left(\frac{\partial p}{\partial \alpha}\right)_\Psi \qquad (4.113)
\end{aligned}
$$

and in the same way for the second equation

$$
-\nabla\alpha \cdot \nabla \times (\nabla\alpha \times \nabla\tilde{\beta}) = 0, \qquad (4.114)
$$

which means nothing else but

$$
(\nabla\alpha \times \mathbf{e}_y) \cdot \nabla B_y = 0 \quad \Longrightarrow \quad B_y = B_y(\alpha). \qquad (4.115)
$$

The formulation of a symmetric equilibrium problem in terms of Euler potentials leads to a set of equations which is far more complicated than the respective Grad-Shafranov equation. However, some boundary conditions can be formulated much more easily when we use Euler potentials. Let us look briefly at the case of a magnetic arcade with prescribed foot point displacement d. As we have seen in Section 3.6 we need to use a nasty integral relation between d and $B_y(A)$ if we use Grad-Shafranov theory. With the Euler potentials, d can be prescribed much easier. Since $\beta(x, y, z) = y + \tilde{\beta}(x, z)$ is constant along field lines, β must have the same value at both foot points of an arcade field line (see Fig. 4.2), i.e.

$$
\beta(x_2, y_2, 0) - \beta(x_1, y_1, 0) = y_2 + \tilde{\beta}(x_2, 0) - y_1 - \tilde{\beta}(x_1, 0) = 0. \qquad (4.116)
$$

From this equation we can derive that

$$d = y_2 - y_1 = -\left(\tilde{\beta}(x_2, 0) - \tilde{\beta}(x_1, 0)\right). \tag{4.117}$$

Thus we can prescribe the footpoint displacement along a field line by imposing the correct boundary conditions on $\tilde{\beta}$!

4.2.2 Laminated Equilibria

A special class of analytical solutions using Euler potentials are the so-called laminated equilibria. They are non-symmetric extensions of the symmetric equilibria with the **B**-field written as

$$\mathbf{B} = \nabla\alpha \times \nabla\beta = \nabla\alpha \times \nabla y \tag{4.118}$$

with $\beta = y$ but $\alpha = \alpha(x, y, z)$. The **B**-field has no y-component and the field lines are confined to planes of constant y. That explains the name 'laminated equilibria'.

The equilibrium equations in this case are given by (neglecting external forces for simplicity)

$$-\Delta_2\alpha = \mu_0 \left(\frac{\partial p}{\partial \alpha}\right)_y \tag{4.119}$$

$$\frac{\partial}{\partial y}\left\{p + \frac{1}{2\mu_0}\left[\left(\frac{\partial \alpha}{\partial x}\right)^2 + \left(\frac{\partial \alpha}{\partial z}\right)^2\right]\right\} = 0 \tag{4.120}$$

with

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}. \tag{4.121}$$

Note that the y-derivative in Eq. (4.120) is the *total* y-derivative, not the y-derivative at constant α !

The second equation simply states that the total pressure $p_T = p + |\mathbf{B}|^2/2\mu_0$ does not vary in the y-direction. Similar equations can be derived for other coordinate systems replacing $\beta = y$ by the appropriate condition.

Solutions for the Cartesian case have been found for

$$p(\alpha, y) = p_1(y)\alpha + p_0(y) \tag{4.122}$$

and

$$p(\alpha, y) = p_2(y)\alpha^2 + p_0(y). \tag{4.123}$$

The most complete discussion has been given by Kaiser and Salat (1997).

Force free laminated solutions can also be found for Cartesian coordinates and spherical coordinates (for a discussion see Low, 1988b,a).

4.3 Other Approaches

As the more or less only systematic 3D equilibrium theory similar to the Grad-Shafranov theory in 2D has so many difficulties, it is natural to try other *ad hoc* approaches to obtain solutions. I will here present one such approach which has been studied intensively by B. C. Low (Low, 1985; Bogdan and Low, 1986; Low, 1991, 1992, 1993a,b) with a few additions by Neukirch (1995, 1997a,b); Neukirch and Rastätter (1999). I will present the theory in Cartesian coordinates with constant external gravitational force. A similar treatment can be carried out in spherical coordinates. I will *not* follow the approach of Neukirch (1995), but that of Neukirch and Rastätter (1999) because it is more transparent, especially after our discussion of linear force-free fields.

The first assumption we make is that the current density $\mathbf{j}$ has the following form

$$\begin{aligned} \mu_0\mathbf{j} &= \alpha\mathbf{B} + \nabla\times(F\nabla\Psi) \\ &= \alpha\mathbf{B} + \nabla(gF)\times\mathbf{e}_z \end{aligned} \tag{4.124}$$

where $\Psi = gz$ has been used, α is assumed to be constant and F is an arbitrary function. As in the linear force-free case, we write the magnetic field $\mathbf{B}$ as

$$\mathbf{B} = \nabla\times[\nabla\times(P\mathbf{e}_z)] + \nabla\times(T\mathbf{e}_z). \tag{4.125}$$

Taking the curl of this expression for $\mathbf{B}$ and equating it to Eq. (4.124), we obtain

$$\begin{aligned} \nabla\times\mathbf{B} &= \nabla\times(-\mathbf{e}_z\Delta P) + \nabla\times[\nabla\times(T\mathbf{e}_z)] \\ &= \nabla\times[\nabla\times(\alpha P\mathbf{e}_z)] + \nabla\times(\alpha T\mathbf{e}_z) + \nabla\times(gF\mathbf{e}_z). \end{aligned} \tag{4.126}$$

We can rearrange the last equation into

$$\nabla\times[-\mathbf{e}_z(\Delta P + \alpha T + gF)] + \nabla\times\{\nabla\times[(T-\alpha P)\mathbf{e}_z]\} = 0. \tag{4.127}$$

Just as in the linear force-free case we can solve this equation by

$$T = \alpha P \tag{4.128}$$

$$\Delta P + \alpha^2 P + gF = 0. \tag{4.129}$$

Obviously, the Helmholtz equation of the linear force-free case has been replaced by a similar equation with one additional term. We now have to be a bit more specific about this term.

Therefore we have a look at the force balance equation. For the Lorentz force term we get

$$\begin{aligned} \mathbf{j}\times\mathbf{B} &= \frac{1}{\mu_0}(\nabla F\times\nabla\Psi)\times\mathbf{B} \\ &= \frac{1}{\mu_0}[(\mathbf{B}\cdot\nabla F)\nabla\Psi - (\mathbf{B}\cdot\nabla\Psi)\nabla F]. \end{aligned} \tag{4.130}$$

Substituting this into the force balance equation results in

$$\frac{1}{\mu_0}(\mathbf{B}\cdot\nabla F-\mu_0\rho)\nabla\Psi-\frac{1}{\mu_0}(\mathbf{B}\cdot\nabla\Psi)\nabla F-\nabla p=0 \tag{4.131}$$

and we conclude that

$$p=p(F,\Psi). \tag{4.132}$$

Using this relation in the force balance equation, we obtain

$$\left(\frac{\partial p}{\partial F}\right)_\Psi = -\frac{1}{\mu_0}\mathbf{B}\cdot\nabla\Psi \tag{4.133}$$

$$\rho = -\left(\frac{\partial p}{\partial \Psi}\right)_F+\frac{1}{\mu_0}\mathbf{B}\cdot\nabla F. \tag{4.134}$$

To be able to get analytical solutions we make a further assumption, namely

$$-\frac{1}{\mu_0}\mathbf{B}\cdot\nabla\Psi=-\frac{1}{\mu_0}\frac{1}{\kappa(\Psi)}F \tag{4.135}$$

equivalent to

$$F=\kappa(\Psi)\mathbf{B}\cdot\nabla\Psi=\frac{1}{g}\xi(z)B_z \tag{4.136}$$

with κ, respectively ξ arbitrary functions. With this assumption Eq. (4.133) becomes

$$\left(\frac{\partial p}{\partial F}\right)_\Psi=-\frac{1}{\mu_0}\frac{1}{\kappa(\Psi)}F \tag{4.137}$$

which we can integrate immediately

$$p=p_0(\Psi)-\frac{1}{2\mu_0\kappa(\Psi)}F^2 \tag{4.138}$$

with $p_0(\Psi)$ an arbitrary background pressure. If we make the further assumption that the density ρ is defined by Eq. (4.134) we have completed the integration of the force balance equation. The temperature T can be calculated if we assume that p, ρ and T are related by the ideal gas law

$$p=\frac{1}{\mu}R\rho T \tag{4.139}$$

where μ is the mean molecular weight.

We still have to determine the magnetic field. To do this we express F in terms of P and substitute this expression for F into Eq. (4.129). We get

$$F=\frac{1}{g}\xi(z)B_z=-\frac{1}{g}\xi(z)\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}\right). \tag{4.140}$$

Substituting this into Eq. (4.129) we obtain

$$\Delta P+\alpha^2 P-\xi(z)\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}\right)=0. \tag{4.141}$$

We have found a linear equation for P and we can solve it with separation of variables. The coefficients of the differential equation depend only on z and we can therefor use Fourier integrals for the x and y dependence

$$P(x, y, z) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \exp(\mathrm{i}k_x x + \mathrm{i}k_y y)\tilde{P}(k_x, k_y, z). \tag{4.142}$$

$\tilde{P}$ has to obey the equation

$$\frac{d^2\tilde{P}}{dz^2} + [\alpha^2 - k^2 + k^2\xi(z)]\tilde{P} = 0 \tag{4.143}$$

with arbitrary $\xi(z)$. This equation has the same mathematical structure as a one-dimensional *Schrödinger equation* where $\xi(z)$ replaces the potential $V(z)$ and $k^2 - \alpha^2$ the energy eigenvalue E. This similarity enables us to find plenty of solutions. For example, $\xi(z) = \xi_0 =$ constant allows the same solution structure as the linear force-free case, $\xi \propto z^2$ gives the "harmonic oscillator" solutions, $\xi \propto 1/z$ the "Coulomb potential" case and so forth.

The solutions derived in this way contain three different contributions to their current density: the linear force-free part $\alpha\mathbf{B}$, a non-linear force-free part given by the parallel component of the $\nabla F \times \nabla\Psi$-term and a non-force-free part given by the perpendicular part of the $\nabla F \times \nabla\Psi$-term.

Due to the similarity of Eq. (4.141) to the Helmholtz equation for P in the linear force-free case, one can develop a Green's function method for this class of MHS solutions in the same way as for the linear force-free solutions (Petrie and Neukirch, 2000). This method can in principle be used to determine not only the magnetic field, but also the plasma pressure, density and temperature from boundary data.

Using the same general approach, solutions have also been found for cases in which F and B_z are nonlinear functions of each other (Neukirch, 1997b).

Bibliography

M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Dover Publications, New York, 1965.

H. Alfvén. *Cosmical Electrodynamics*. Clarendon Press, Oxford, 1950.

T. Amari, J. J. Aly, J. F. Luciani, T. F. Boulmezaoud, and Z. Mikic. Reconstructing the solar coronal magnetic field as a force-free magnetic field. *Solar Physics*, 174:129–149, 1997.

R. Balescu. *Plasma Transport Theory*, volume 1. North-Holland, 1988.

W. H. Bennett. Magnetically self-focusing streams. *Physical Review*, 45: 890–897, 1934.

T. J. Bogdan and B. C. Low. The three-dimensional structure of magnetostatic atmospheres. II - Modeling the large-scale corona. *Astrophysical Journal*, 306:271–283, July 1986.

Y. T. Chiu and H. H. Hilton. Exact Green's function method of solar force-free magnetic-field computations with constant alpha. I - Theory and basic test cases. *Astrophysical Journal*, 212:873–885, March 1977.

T. G. Cowling. *Magnetohydrodynamics*. Adam Hilger, London, 1976.

J. W. Edenstrasser. Unified treatment of symmetric MHD equilibria. *Journal of Plasma Physics*, 24:299–313, 1980a.

J. W. Edenstrasser. The only three classes of symmetric MHD equilibria. *Journal of Plasma Physics*, 24:515–518, 1980b.

V. M. Fadeev, I. F. Kvartskhava, and N. N. Komarov. Self-focusing of local plasma currents. *Nuclear Fusion*, 5:202–209, 1965.

T. Gold and F. Hoyle. The origin of solar flares. *Monthly Notices of the Royal Astronomical Society*, 120:89–105, 1960.

H. Grad. Theory and applications of the nonexistence of simple toroidal plasma equilibrium. *International Journal of Fusion Energy*, 3:33–46, 1985.

E. G. Harris. On a plasma sheath separating regions of oppositely directed magnetic field. *Nuovo Cimento*, 23:115–, 1962.

J. R. Hundhausen, A. R. Hundhausen, and E. G. Zweibel. Magnetostatic atmosphere in a spherical geometry and their application to the solar corona. *Journal of Geophysical Research*, 86:11117–11126, 1981.

J. D. Jackson. *Classical Electrodynamics*. John Wiley & Sons, New York, 2nd edition, 1975.

R. Kaiser and A. Salat. New classes of three-dimensional ideal-mhd equilibria. *Journal of Plasma Physics*, 57:425–448, 1997.

R. Kippenhahn and A. Schlüter. Eine Theorie der solaren Filamente. *Zeitschrift für Astrophysik*, 43:36–63, 1957.

J. Liouville. Sur l'équation aux différences partielles $\partial^2 \log \lambda / \partial u \partial v \pm \lambda / 2a^2 = 0$. *Journal de Mathématiques Pures at Appliquées*, 18:71–72, 1853.

R. M. Lothian and P. K. Browning. Coronal magnetic field equilibrium with discrete flux sources. *Solar Physics*, 161:289–316, 1995.

B. C. Low. Three-dimensional structures of magnetostatic atmospheres. I - Theory. *Astrophysical Journal*, 293:31–43, June 1985.

B. C. Low. Magnetic field configurations associated with polarity intrusion in a solar active region. *Solar Physics*, 115:269–276, 1988a.

B. C. Low. On the hydromagnetic stability of a class of laminated force-free magnetic fields. *Astrophysical Journal*, 330:992–996, July 1988b.

B. C. Low. Three-dimensional structures of magnetostatic atmospheres. III - A general formulation. *Astrophysical Journal*, 370:427–434, March 1991.

B. C. Low. Three-dimensional structures of magnetostatic atmospheres. IV - Magnetic structures over a solar active region. *Astrophysical Journal*, 399:300–312, November 1992.

B. C. Low. Three-dimensional structures of magnetostatic atmospheres. V - Coupled electric current systems. *Astrophysical Journal*, 408:689–692, May 1993a.

B. C. Low. Three-dimensional structures of magnetostatic atmospheres. VI - Examples of coupled electric current systems. *Astrophysical Journal*, 408:693–706, May 1993b.

B. C. Low, A. J. Hundhausen, and E. G. Zweibel. Nonlinear periodic solutions for the isothermal magnetostatic atmosphere. *Physics of Fluids*, 26: 2731–2741, 1983.

B. C. Low and Y.-Q. Lou. Modeling solar force-free magnetic fields. *Astrophysical Journal*, 352:343–352, March 1990.

L. Mestel. *Stellar Magnetism*. Clarendon Press, Oxford, 1999.

P. M. Morse and H. Feshbach. *Methods of Theoretical Physics*, volume 1. McGraw-Hill, New York, 1953a.

P. M. Morse and H. Feshbach. *Methods of Theoretical Physics*, volume 2. McGraw-Hill, New York, 1953b.

T. Neukirch. On self-consistent three-dimensional solutions of the magnetohydrostatic equations. *Astronomy and Astrophysics*, 301:628–640, 1995.

T. Neukirch. 3D solar magnetohydrostatic structures. *Physics and Chemistry of the Earth*, 22:405–409, 1997a.

T. Neukirch. Nonlinear self-consistent three-dimensional arcade-like solutions of the magnetohydrostatic equations. *Astronomy and Astrophysics*, 325:847–856, 1997b.

T. Neukirch and L. Rastätter. A new method for calculating a special class of self-consistent three-dimensional magnetoshydrostatic equilibria. *Astronomy and Astrophysics*, 348:1000–1004, 1999.

E. N. Parker. *Cosmical Magnetic Fields*. Clarendon Press, Oxford, 1979.

G. J. D. Petrie and T. Neukirch. The Green's function method for non-force-free three-dimensional solutions of the magnetohydrostatic equations. *Astronomy and Astrophysics*, 356:735–746, 2000.

D. Pfirsch. Mikroinstabilitäten vom Spiegeltyp in inhomogenen Plasmen. *Zeitschrift für Naturforschung*, 17a:861–870, 1962.

E. R. Priest. *Solar Magnetohydrodynamics*. D. Reidel, Dordrecht, 1982.

P. H. Roberts. *An Introduction to Magnetohydrodynamics*. Longmans, London, 1967.

R. Rosner, B. C. Low, K. Tsinganos, and M. A. Berger. On the relationshsip between the topology of magnetic field lines and flux surfaces. *Geophysical and Astrophysical Fluid Dynamics*, 48:251–271, 1989.

E. Schatzmann. Model of force free fields. In R. Lüst, editor, *Stellar and Solar Magnetic Fields*, volume 22 of *IAU Symposium*, page 337, Amsterdam, 1965. International Astronomical Union, North-Holland.

K. Schindler and J. Birn. Magnetotail theory. *Space Science Reviews*, 44: 307–355, 1986.

J. Schmid-Burgk. Zweidimensionale selbstkonsistente Lösungen der stationären Wlassowgleichung für Zweikomponententplasmen. Master's thesis, Ludwig-Maximilians-Universität München, 1965.

L. S. Solov'ev. Symmetric magnetohydrodynamic flow and helical waves in a circular plasma cylinder. In M. A. Leontovich, editor, *Reviews of Plasma Physics*, volume 3, pages 277–325. Consultants Bureau, New York, 1967.

L. S. Solov'ev. The theory of hydromagnetic stability of toroidal plasma configurations. *Soviet Physics JETP*, 26:400–407, 1968.

P. A. Sturrock. *Plasma Physics*. Cambridge University Press, Cambridge, Endgland, 1994.

M. S. Wheatland. A better linear force-free field. *Astrophysical Journal*, 518:948–953, 1999.

E. G. Zweibel and A. J. Hundhausen. Magnetostatic atmospheres: A family of isothermal solutions. *Solar Physics*, 76:261–299, 1982.