Applying the Cramer-Rao Lower Bound to Spectroscopic Measurements

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Introduction

Spectrometers experiments are essential in astronomy and provide us with much useful information on the emitting plasma. In solar physics, emission spectra are commonly modeled as Gaussian emission lines plus a background. It is clear that the construction of the experiment and the details of how observations are made influences the quality of the information derivable. A ***theoretical framework*** discussing the ***precision*** with which an experiment can measure the emission line variables can be created by linking the notion of ***experimental precision*** to the ***Cramer-Rao lower bound*** (Kendall and Stuart, 1960*) via ***models of spectrometers and line emission.***

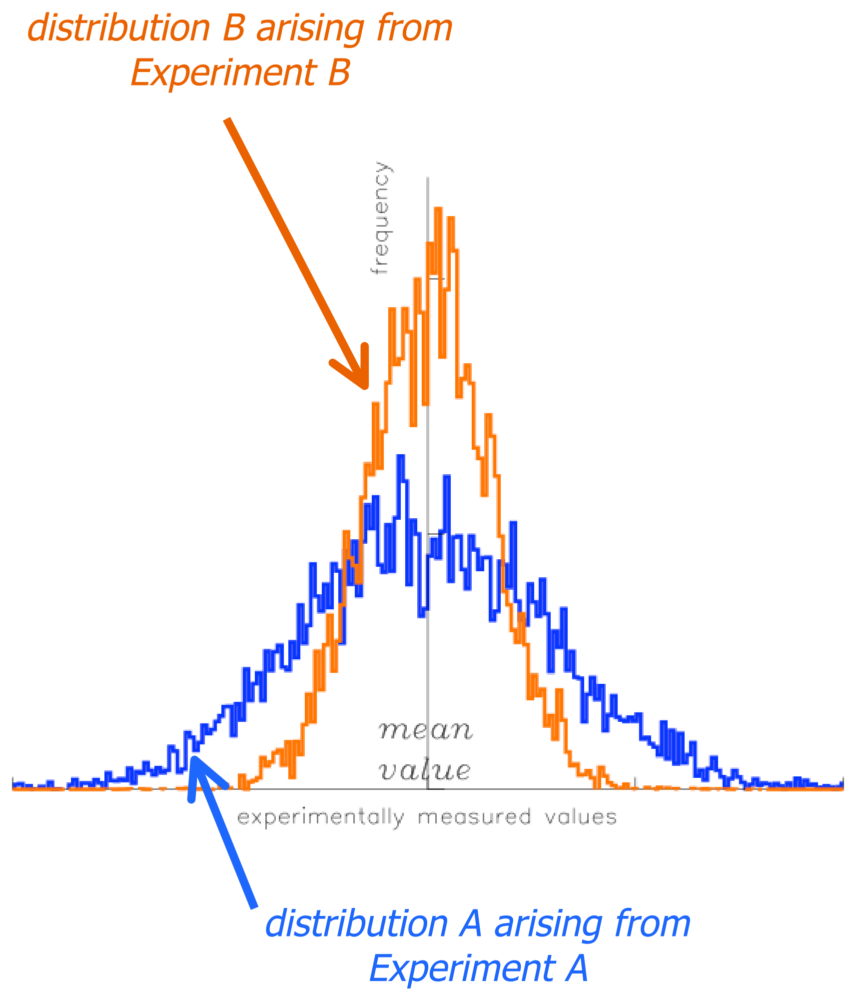
** Kendall, M. G., Stuart, A.: The Advanced Theory of Statistics (vol. 2), 3rd edition, Charles Griffin and Co., Ltd., 1960.*

Experimental Precision

An experiment is ***precise*** if the distribution of values around the mean value is ***small****. In the figure across, the ***sample variance*** of distribution A is smaller than the sample variance of distribution B. Hence we can say that Experiment A is more precise than Experiment B

The ***Cramer-Rao lower bound*** defines a lower limit to the sample variance - and hence a ***maximum precision*** - that a measured parameter can attain in the presence of random measurement errors.

This definition of precision makes no comment on experimental ***accuracy***; an experiment is accurate if the mean observed value is “close” to the true value. Systematic errors can affect the experimental accuracy. In this study only precision and random measurement errors are considered.



**Barford, N.C.:Experimental measurements: Precision, Error and Truth, 2nd Edition, John Wiley and Sons, 1985.*

Cramer-Rao Lower Bound

In a model spectrometer, the data D_i, $1 \leq i \leq N$, (N pixels across the CCD) measured in each pixel channel follows a normalized distribution function

$$f_i(D_i, V)$$

where V is the set of variables describing the data. The overall probability distribution function is

$$L = \prod_{i=1}^{N} f_i(D_i, V)$$

Maximizing *L* as a function of V defines $\hat{V}$

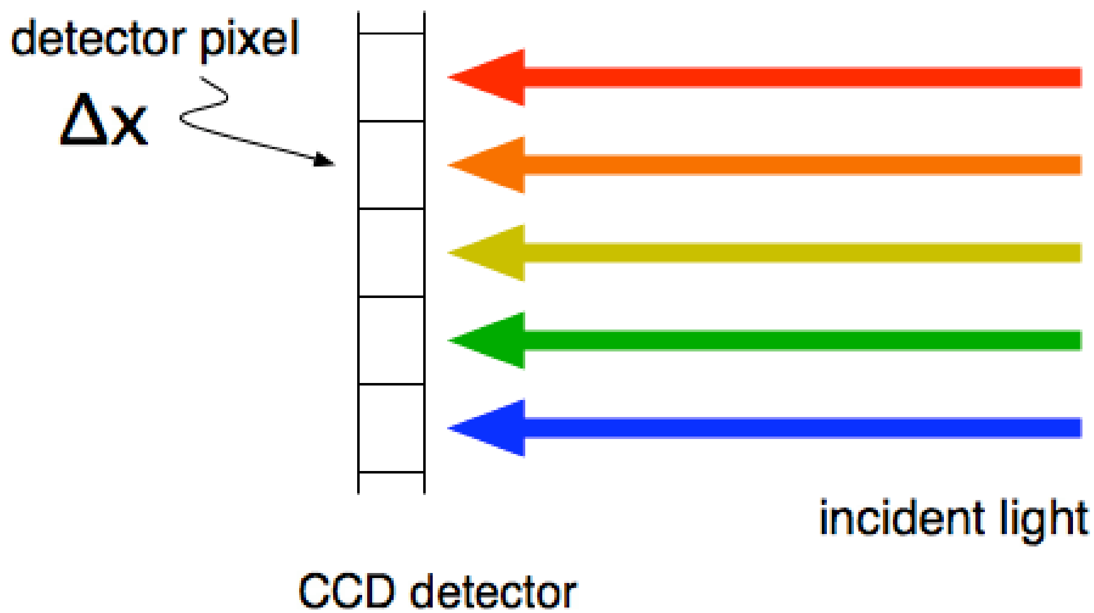
the maximum likelihood solution. For *t(V)* (with some restrictions on *t*), it can be shown that

$$\mathrm{var}(t) \geq \sum_{p=1}^{N_v} \sum_{q=1}^{N_v} \frac{\partial t}{\partial V_p} \frac{\partial t}{\partial V_q} I^{-1}$$

where $$I_{pq} = Expectation\left(\frac{\partial^2 \log L}{\partial V_p \partial V_q} \right) \qquad (1)$$

is the information matrix for variables V_p,V_q, $1 \leq p,q \leq N_v$, evaluated at $\hat{V}$. If *t* is set to one of the variables in the model then for V= $\hat{V}$ we obtain ***a lower limit to the sample variance that measurements of that variable can achieve.*** Details about the spectrometer model and line emission profile are contained in *L*.

A model CCD spectrometer



Light is incident on a CCD array, each pixel having size Δx. No pixel crosstalk occurs.

Emission model

- Solar emission lines are modeled as Gaussians of amplitude A_i (photons/px), width σ_I (px) centered on c_i (px), n_G lines.
- Background emission is a constant μ (photons/px).
- The emission follows Poisson statistics

$$emission(x) = \mu + \sum_{i=1}^{n_G} A_i \exp\left[\frac{(x - c_i)^2}{2\sigma_i^2}\right] \quad (2)$$

This completes the necessary parts to the theoretical framework - we can now pose questions about the theoretical limits to the emission line variables' sample variances, i.e., ***the best precision attainable***, in different contexts.

Application 1: *how many pixels across the line?*

To illustrate some applications of this framework, a ***CDS*-like*** spectrometer observing the coronal Fe XVI 360.76Å is considered. ***Note that these applications are not intended to simulate CDS in detail***. Rather, they are intended to demonstrate the types of questions that can be posed by a Cramer-Rao approach to ***general spectrometer observations.*** In its synoptic studies, CDS returns an 18 pixel window containing the Fe XVI 360.76Å line - typical values for this line are shown in the table across and are used in this and the following two example applications.

How many pixels across the line are needed in order to precisely measure the line? Increasing the window width will yield more information on the background and so make it easier to distinguish the background emission from the emission line - assuming the line is centered in the window, ***how many pixels are sufficient to do this?***

Fe XVI line

$A_1 = 1434\gamma/px$

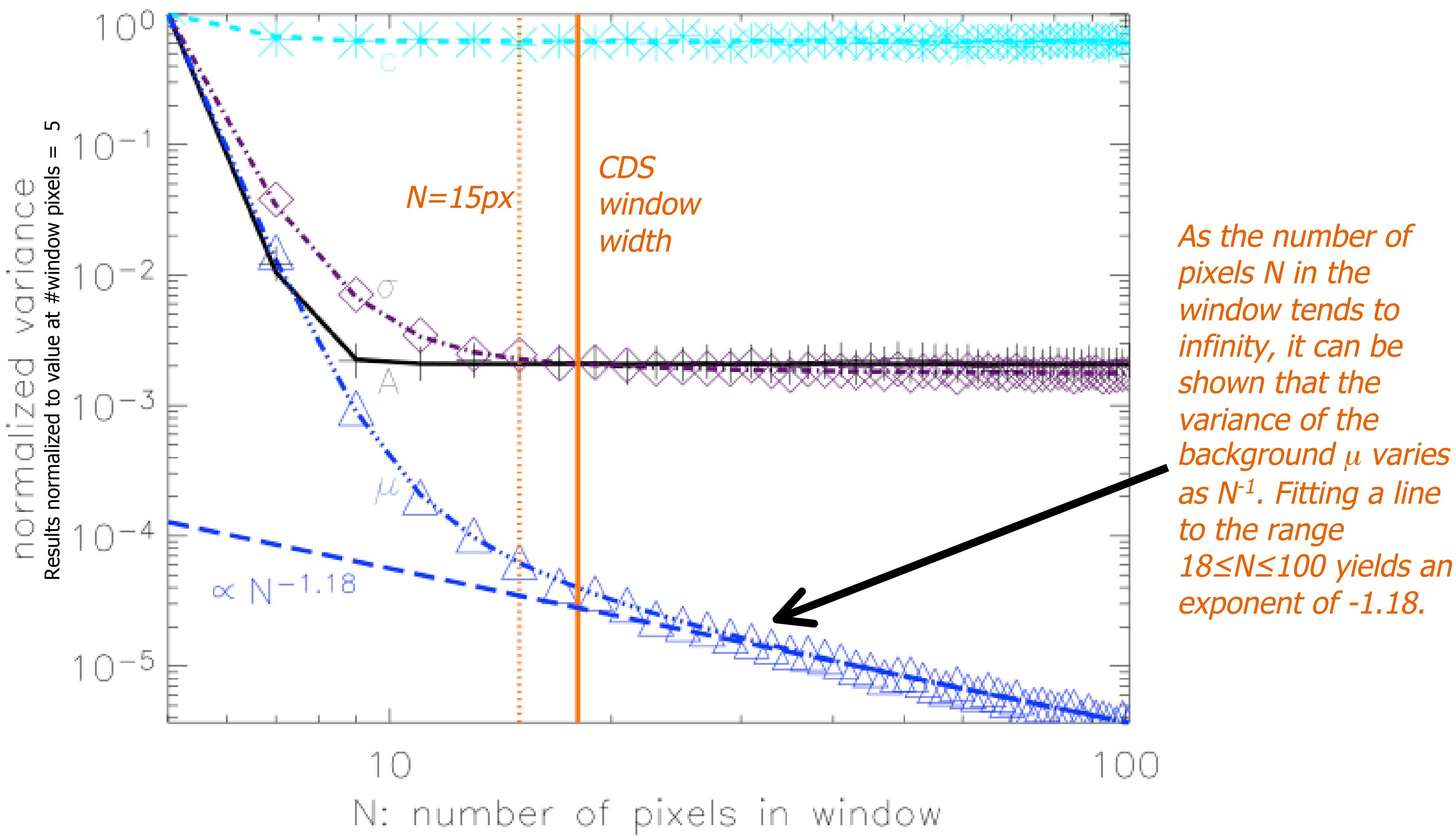
$c_1 = 0.0px$

$\sigma_1 = 1.62px$

$\mu = 114\gamma/px$

18 pixels in spectral window

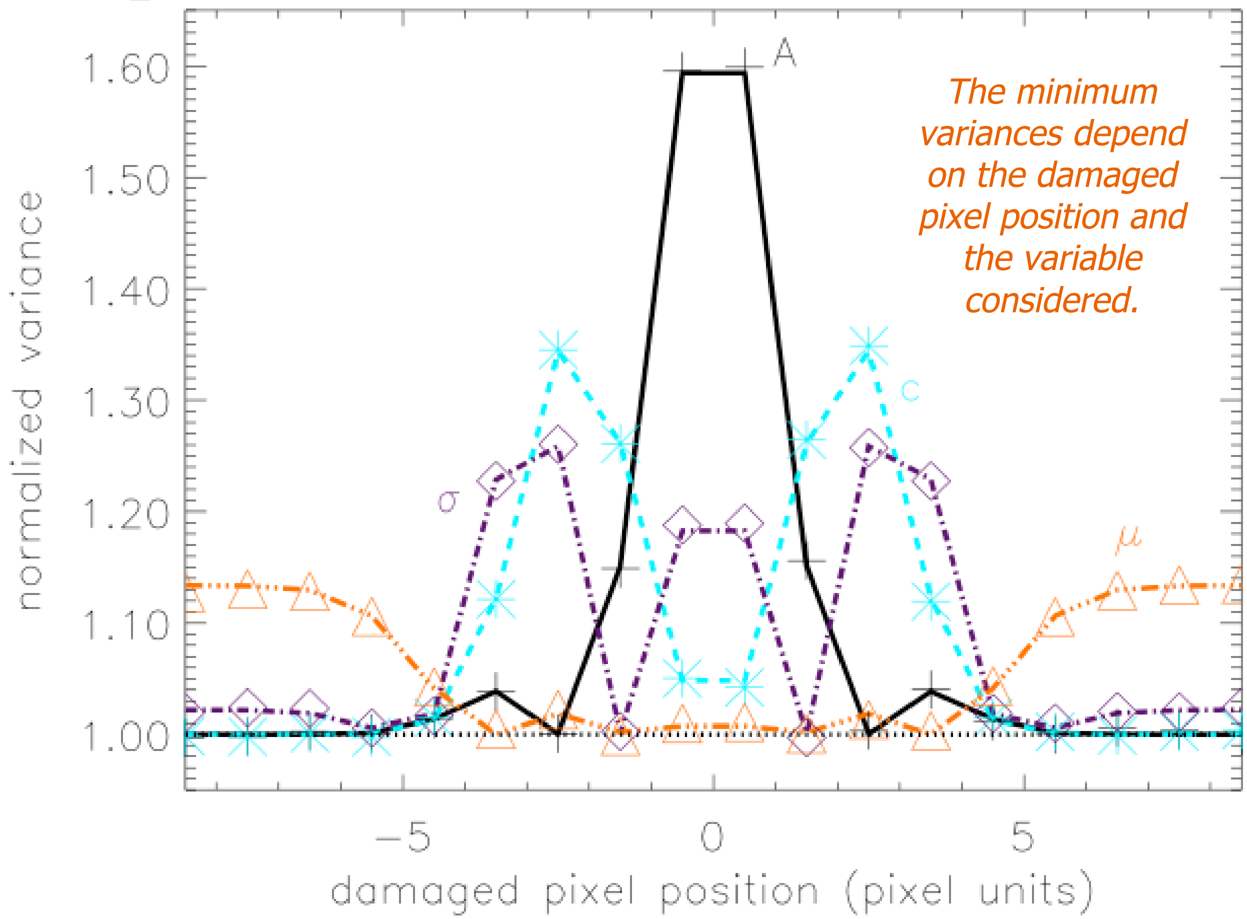
**Coronal Diagnostic Spectrometer, onboard SOHO, the Solar and Heliospheric Observatory.*



The figure above shows how the Cramer-Rao minimum variances (plot lines) in each model variable change with the width of the spectral window. The behavior is also confirmed by simulation (plot symbols, derived by fitting Equation 2 to simulated data and calculating the sample variances of the fit variables). It is clear that beyond about 15 pixels widening the window results in ***very little gain*** in precision in the Gaussian variables A, c and σ. Such a study can be used to define the ***minimum desirable window size*** for an observation.

Application 2: *one damaged pixel*

Assume that one pixel in the 18 pixel spectral window of Fe XVI 360.76Å is destroyed by a particle hit. The pixel does not contribute to the information matrix (Eqn. 1), and so measurements of the model variables must be less precise. The figure opposite shows the effect (as a function of pixel position) on Cramer-Rao minimum variances (plot lines) of removing one pixel from the spectrum. The behavior is also confirmed by simulation (plot symbols, derived by fitting Equation 2 to simulated data and calculating the sample variances of the fit variables). ***Such a study can be used to help optimize observations from damaged instruments.***



Results are normalized to Cramer-Rao minimum variances calculated when all pixels contribute to the information matrix. Removing a pixel always increases the minimum variances, making the experiment less precise.

Application 3: *optimising for total line intensity*

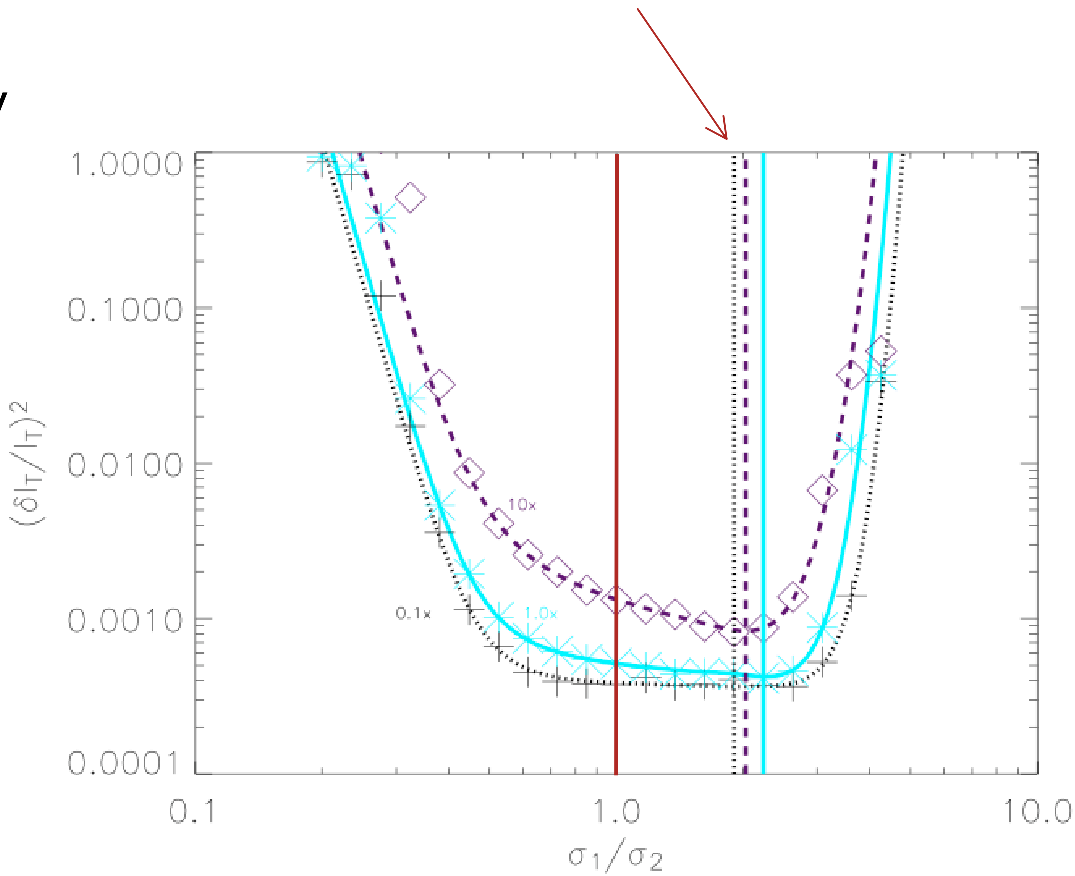
Let us assume that we want to measure the total integrated intensity of a Gaussian emission line, given by

$$I_T = \sqrt{2\pi}\, A\sigma$$

How should the line be presented to the detector in order to make the experiment as precise as possible? There are two competing factors: narrowing the line will improve the signal to noise ratio in each pixel, but we will lose information on the width. Conversely, a wider line presented to the detector would improve our knowledge of the line width at the expense of degrading the signal to noise ratio in each pixel. There should exist a balance between these effects which ***optimizes*** the precision of the experiment with respect to measuring the total intensity.

The figure opposite shows the dependence of $(\partial I_T/I_T)^2$ - the relative variance of the total intensity - as a function of line width σ_2 presented to the detector (measured relative to σ_1) for three different emission profiles differentiated by their background emission- 0.1μ, μ and 10μ. For each value of σ_1/σ_2 the number of photons in the emission line is constant. Minima in the Cramer-Rao calculated values of $(\partial I_T/I_T)^2$ are found (plot lines) and are confirmed via simulation (plot symbols, derived by fitting Equation 2 to simulated data and calculating the sample variances of the fit variables). At this value of width the variance $(\partial I_T/I_T)^2$ is minimized - ***the experiment is as precise as possible***.

positions of minima



$(\partial I_T/I_T)^2$ at $\sigma_1/\sigma_2 = 1$ is 1.05, 1.21 and 1.60 times the value at minimum for background emissions 0.1μ, μ and 10μ respectively.

Conclusions and future work

- A ***theoretical framework*** has been developed in which questions about spectrometer precision limits in a variety of different situations can be posed.
- Studies of precision can be used to ***aid the design*** of spectrometers and the construction/operation of observational studies.
- There are many possible extensions to this work. For example:
  - the effect on precisions of multiple Gaussians, different backgrounds.
  - inclusion of pixel/CCD crosstalk effects.
  - different model spectrometers, non-Gaussian line emission models, more accurate statistical models of noise in spectrometers.