Flux tube disconnection: An example of three-dimensional reconnection

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Magnetic reconnection in three dimensions (3D) is fundamentally different in many respects from its two-dimensional counterpart. An analytical example of a global 3D general magnetic reconnection process is presented here, in which a magnetic flux tube has its footpoints rotated in different directions. A localized nonideal region is found in the center of the tube where the field lines are twisted as a result of the footpoint counter-rotation. A qualitative analysis demonstrates how the parameters that determine the strength and geometry of the magnetic field depend on the rotational driving velocity. The reconnection rate measures the rate at which field lines within the entire flux tube are changing their magnetic connections, and is shown to be proportional to the driving velocity imposed on its footpoints. The qualitative estimates are confirmed by an exact kinematic 3D magnetohydrodynamic analytical solution, which possesses a localized three-dimensional current and determines the reconnection rate quantitatively. © 2007 American Institute of Physics. [DOI: 10.1063/1.2783257]

I. INTRODUCTION

Magnetic reconnection is a fundamental process that is responsible for a wide range of dynamic events in astrophysical, space, and laboratory plasmas. Reconnection allows for a change in topology of an otherwise ideal plasma and can result in substantial energy release—it was first suggested as a mechanism for particle acceleration in solar flares.

The first models for reconnection were two-dimensional (2D) [i.e., the flow and field are confined to the ($x,y$) plane and $\partial / \partial z=0$]. Under these circumstances, reconnection can only occur at an X-type null point of the field, with the rate at which it occurs being given by the electric field at the X-point and measuring the amount of flux being cut and rejoined. In 2D models, magnetic flux is transported by a plasma flow toward the null point, where it is reconnected and, subsequently, transported away from the null point at a relatively high speed.

In the first quantitative model for the process, the Sweet-Parker model, a current sheet with length equal to the global length scale is contained within oppositely directed magnetic fields. The reconnection rate is proportional to $S^{-1/2}$ (where $S$ is the Lundquist number) and, so, with the high values of $S$ in space and astrophysical plasmas, it is too slow to account for observed events such as solar flares. To overcome this problem, the Petschek model takes a much shorter current sheet with a standing slow shock at each of its corners and achieves a reconnection rate dependent on $1/\ln S$. In both the Sweet-Parker and Petschek models, magnetic energy is converted to thermal energy via Ohmic heating and also to kinetic energy of the plasma, with most of the conversion in the Petschek process taking place at the standing shocks.

Reconnection models that scale as the Sweet-Parker rate (or slower) are described as slow while those that scale faster than $S^{-1/2}$ (including the Petschek model) are described as fast. We now understand that the Petschek model is just one of a much wider class of fast reconnection solutions that includes families of almost uniform and nonuniform regimes; which solution is achieved in numerical simulations depends on the boundary conditions imposed. Petschek reconnection itself appears to also require a nonuniform localized resistivity to be imposed. However, localized regions of enhanced resistivity are expected to occur in practice (since intense current concentrations are likely to drive microinstabilities that themselves enhance the resistivity) and so this requirement is not thought to be unreasonable. Reconnection may also arise as a result of an initial localized instability, such as the tearing mode, and in these situations the dynamics are not so sensitive to the choice of boundary conditions. Local plasma conditions, such as the presence of an anomalous resistivity, may also play an important role in determining the resulting reconnection configuration.

For a review of 2D reconnection, see, for example, Ref. 5.

Moving into 2.5D and 3D configurations, reconnection events that exhibit several similarities to the 2D case have been modeled. For example, Ugai and Zheng found a 3D fast reconnection configuration including standing slow shocks to be set up when a small disturbance together with an enhanced resistivity was imposed on a current sheet. A 3D configuration with Petschek-like properties, including slow shocks, was also obtained by Linton and Longcope after they imposed a short burst of reconnection in a localized sphere. Several 3D effects were also observed in this case, with the shock geometry being substantially modified. In a 2.5D study of the interaction of two emerging flux systems, Archontis et al. found Petschek-like reconnection with slow-mode shocks extending from the current sheet that formed at the interface between the two flux systems.
FIG. 1. Classification of breakdown of magnetic connection according to Ref. 16. Various branches of general magnetic reconnection are shown. Reconnection is classified as global when a change in magnetic connectivity occurs for plasma elements that do not themselves pass through the nonideal region.

nature of the reconnection remained the same for both uniform and nonuniform examples of resistivity.

However, not all 3D reconnection configurations show similarities to the 2D case and three-dimensional field configurations result in a much wider class of reconnection solutions. Indeed, while several features of 2D reconnection models, such as a plasma flow across the separatrices of the field, or a normal electric field component at the X-point, were proposed as definitions for reconnection, it was not until more general, 3D, circumstances were considered that the appropriateness of each of these conditions became clear. In Schindler et al., the applicability of these ideas to 3D geometries was considered. The authors argued that any general definition should be structurally stable, i.e., not depend on small modifications to the system under consideration. They proposed that a definition, first considered by Axford, based on a change of connectivity of plasma elements due to a localized violation of the frozen-in field condition, should be used. This forms the basis of general magnetic reconnection (GMR). The condition

\[ \int E_{id} dl \neq 0, \]

evaluated along a field line, is required for GMR, being a generalization to 3D of the 2D normal electric field component at an X-point. In order to distinguish between reconnection and diffusive processes, the additional requirement that the nonideal term in Ohm’s law (which itself is the cause of the breakdown of the frozen-in condition) be localized is also imposed. Equivalently, GMR applies only to situations in which the global magnetic Reynolds number \( R_{mg} \) is large, i.e., \( R_{mg} \gg 1 \). Figure 1 illustrates the regimes of breakdown of magnetic connection and so also the branches of general magnetic reconnection (see Ref. 16). The breakdown of magnetic connection may be caused by a resistive term in Ohm’s law, but also by other nonideal terms, such as the pressure tensor.

In 3D, there are several forms of reconnection associated with magnetic null points. Magnetic separators, field lines connecting two null points and therefore lying at the interface between four distinct flux domains, are thought to be favorable locations for current buildup. Separator reconnection occurs along separators connecting null points and has been observed in many numerical experiments (see, e.g., Refs. 23–26). Reconnection may also be linked to the spine and fan associated with null points, depending on the direction of the current.

Finite-B reconnection assumes the magnetic field does not vanish within the nonideal region, and may be classified further as local or global according to how the change in connectivity arises. If a change in magnetic connectivity occurs for plasma elements that do not themselves pass through the nonideal region, then the process is global; otherwise it is local. The change in connectivity necessarily involves plasma elements located on different sides of the nonideal region, and reconnection occurs only for elements which are, at some time, connected to the nonideal region.

The complexity of 3D reconnection has meant that much of the analytical work so far has focused on the kinematic problem, where the implications of the momentum equation are ignored and the effects of Ohm’s law, together with Maxwell’s equations and the continuity equation are considered (see Ref. 29 and references therein). In these studies, a particular magnetic field configuration is imposed from which the plasma velocity and electric field are deduced. The simplified nature of the analysis allows an isolated nonideal region to be taken (i.e., bounded in all three dimensions). Such a localized nonideal region is the generic situation in astrophysical plasmas. The models have identified several new features of 3D reconnection. For example, under certain circumstances, reconnection solutions may be decomposed into ideal and nonideal parts according to

\[ E_{\text{nonid}} + v_{\text{nonid}} \times B = \frac{1}{\sigma} j, \]

\[ E_{\text{id}} + v_{\text{id}} \times B = 0, \]
i.e., there is no longer a direct coupling between particular nonideal reconnective solutions and ideal flows. The localization of the nonideal region implies the existence of counter-rotating plasma flows within the flux tube that consists of all the field lines threading the nonideal region. Both of these new features have subsequently been backed up by dynamic analytical modeling as well as by 3D magnetohydrodynamic (MHD) simulations.

Reconnection models in which the magnetic field has an O-type topology have received little attention thus far. Recently, however, Ref. 32 presented numerical simulations of 3D reconnection due to rotational motion of the footpoints of magnetic flux tubes. In their simulations, stagnation flows were observed, together with an X-type current structure, although the magnetic field showed an O-type configuration in cross-sectional planes. Here we present an example of 3D reconnection in an elliptic geometry that allows for analyti-
cal modeling, namely the reconnection that occurs as a result of the counter-rotation of the ends of a magnetic flux tube, which we refer to as flux-tube disconnection. A qualitative description of the process is outlined in the next section, where we also demonstrate how it fits into the theme of GMR but looks very different from previously considered reconnection models. A more detailed quantitative description is given in Sec. III.

II. A QUALITATIVE MODEL FOR FLUX TUBE DISCONNECTION

Consider a steady-state situation (Fig. 2) in which a magnetic flux tube, of radius $a$, has footpoints located at $z = \pm H$ and a typical vertical magnetic field strength $b_0$. Assume that the two footpoint ends are being rotated in different directions with speed $v_0$, and that, as a result of the counter-rotation, a twist in the field is produced within the flux tube in the shaded region, i.e., between $z = \pm L$. That is, assume that within the nonideal region, the magnetic field may be written as

$$B = B_\theta \hat{\theta} + B_z \hat{z} = k b_0 \hat{\theta} + b_0 \hat{z},$$

where $k$ is a constant indicating the order of magnitude ratio of the toroidal field strength to the guide field. The flow is assumed to be incompressible, which is satisfied automatically for a purely azimuthal plasma velocity that is independent of $\theta$.

The localized poloidal field component results in a current, and therefore a nonideal region, that is localized in all three dimensions. The current has a component parallel to the magnetic field, and does not close within the nonideal region; we assume that the return current is diffuse and is spread over a sufficiently large volume that its local effect may be neglected. A sketch of the current structure is shown in Fig. 3.

Several questions now arise in the analysis. How, in an order of magnitude sense, is the twist in magnetic field related to the driving plasma velocity $v_0$? Specifically, how do the parameters $L$ and $k$, which determine the extent of the nonideal region and the strength of the poloidal field, respectively, depend on $v_0$? Which parameters determine the rate of reconnection? How is the rate of reconnection dependent on the driving velocity $v_0$?

We can gain an insight into these questions by examining Ohm’s law and Faraday’s law for a steady state,

$$E + v \times B = \frac{1}{\sigma} j,$$

$$\nabla \times E = 0,$$

where $\sigma$ is the electrical conductivity. Consider the ideal region above and below $z = \pm L$. There, $j = 0$, and Ohm’s law reduces to

$$E_c + v_0 b_0 = 0,$$

where $E_c$ is the typical radial electric field. Similarly, along the central axis ($r = 0$), the plasma velocity is zero by symmetry and

$$E_c = \frac{1}{\sigma} j_z.$$

Now, $\nabla \times B = \mu j$ implies that along the axis we also have

$$j_z = \frac{1}{\mu} \left( \frac{B_\theta}{r} + \frac{\partial B_z}{\partial r} \right) = \frac{2 k b_0}{\mu a}.$$

Next consider a loop integral, as illustrated in Fig. 4, consisting of the part of the central field line from below to above the nonideal region, a field line on the boundary of the nonideal region, and two connecting radial lines. Integrating around this loop and using Faraday’s law and using the symmetry of solutions above and below the $z$ axis gives
We may also make an estimate of the rate of reconnection \((d\Phi_{\text{rec}}/dt)\), i.e., the rate of change of magnetic flux. This rate is given by the integral of the parallel electric field along the reconnection line, i.e., along the central axis,

\[
\frac{d\Phi_{\text{rec}}}{dt} = 2 \frac{a}{H} v_0 \tag{6}
\]

where expressions (1)–(3) have been used. This shows us that the rate of reconnection is proportional to the rotational driving velocity. The reconnection rate may be interpreted as the rate at which all the field lines within the flux tube are changing their connections, and so this estimate of the reconnection rate agrees with our intuitive understanding of the process. In dimensionless terms, we have

\[
\frac{d\Phi_{\text{rec}}}{dt} = \frac{2 a v_0}{H v_A} \tag{7}
\]

Thus the dimensionless reconnection rate is proportional to the Alfvén Mach number \((v_0/v_A)\) of the driving flow and the ratio \((a/H)\) of the nonideal region radius \((a)\) to the ambient scale height \((H)\). Thus, if \(a/H\) and \(v_0/v_A\) are significant fractions of unity, we have fast reconnection, i.e., at a significant fraction of the Alfvén speed; otherwise the reconnection is slow (see Sec. IV).

In the next section, we present an analytical model for the reconnection process described above. We show that the exact kinematic solution obtained there agrees with the intuitive understanding developed in this section.

### III. KINEMATIC MODEL

We present here an axisymmetric analytical model for flux tube disconnection. The model is kinematic (i.e., the equation of motion is neglected) and stationary, and satisfies the following equations:

\[
E + v \times B = \frac{1}{\sigma} j, \tag{8}
\]

\[
\nabla \times E = 0, \tag{9}
\]

\[
\nabla \times B = \mu j, \tag{9}
\]

\[
\nabla \cdot B = 0, \tag{9}
\]

\[
\nabla \cdot v = 0. \tag{9}
\]

Several of the previous 3D analytical models of reconnection that have helped to increase our understanding of the process have been kinematic.\(^{33,29,28}\) Neglecting the momentum equation allows for analytical progress to be made and, as a result, several new features of 3D reconnection have been found, such as counter-rotational plasma flows, that are not present in 2D. The features are expected to be inherent to the 3D reconnection processes, and not consequences of considering only a subset of the MHD equations. Indeed, using kinematic models as a guide, subsequent numerical experiments (such as that of Ref. 31) and analytical models with
the momentum equation included\textsuperscript{30} have investigated the suggested features and confirmed their importance in the 3D process.

A typical feature of reconnection in astrophysical plasmas is that nonideal regions are localized in 3D as a result of intense current concentration. In such regions, the resistivity is expected to be enhanced by current-driven microinstabilities. We therefore consider here a magnetic field that leads to a localized current, and additionally impose a localization of the resistivity. This is one of the features that distinguishes our model from previous kinematic models where a localized nonideal region was obtained through an enhancement of the resistivity alone.

Working throughout in cylindrical coordinates \((r, \theta, z)\), the magnetic field is prescribed as

\[
\mathbf{B} = 2b\frac{k}{a} r \exp \left( -\frac{r^2}{a^2} - \frac{z^2}{L^2} \right) \hat{\theta} + b_0 \hat{z}.
\]

The azimuthal component of the magnetic field is localized and so generates a twist in the magnetic field close to the origin, while at large distances the field is uniform in the \(z\) direction. The width of the flux tube is dependent on the parameter \(a\), the extent of the twist in the \(z\) direction on the parameter \(L\), and the ratio of the toroidal field to the guide (or axial) field given by the parameter \(k\). Some typical field lines are illustrated in Fig. 5; note that each field line remains on a surface of constant radius.

This simple field configuration allows for a direct integration to find the equations of the field lines. This proves to be useful later, where we integrate along the field lines in order to obtain expressions for the remaining terms. The equations, \(\mathbf{R}(r_0, \theta_0, s)\), of the field lines passing through the point \((r_0, \theta_0, z=0)\) are given by

\[
R = r_0,
\]

\[
\Theta = 2b\frac{k}{a} r_0 \exp \left( -\frac{r_0^2}{a^2} - \frac{b_0^2 z^2}{L^2} \right),
\]

\[
Z = b_0 s.
\]

The inverse mapping, \(\mathbf{R}(r, \theta, s)\), is given by

\[
R_0 = r,
\]

\[
\Theta_0 = 2kb \frac{r}{a} \exp \left( -\frac{r^2}{a^2} - \frac{b_0^2 z^2}{L^2} \right),
\]

\[
Z_0 = -b_0 s.
\]

The twist of the flux tube generates closed poloidal rings of current, given by

\[
\mathbf{j} = \frac{4b_0 k}{\mu a} \exp \left( -\frac{r^2}{a^2} - \frac{z^2}{L^2} \right) \left( \frac{r z}{L^2} + \frac{a^2 - r^2}{a^2} \frac{z}{L} \right).\]

An example is shown in Fig. 6, where the current is seen to be localized in three dimensions around the origin, and has its greatest strength along the axis of the flux tube. Now the nonideal term, \(1/\sigma j\), in Ohm's law [Eq. (8)] would already be localized given a uniform electrical conductivity, \(\sigma\). Here, however, we choose to impose in addition a form for \(1/\sigma\) leading to a localized magnetic diffusivity \(\eta\). We take

\[
\eta = \frac{1}{\sigma_0 \mu} \exp \left( -\frac{4r^2}{a^2} - \frac{z^2}{L^2} \right).
\]

Regions of intense current concentration are expected to give rise to enhanced diffusivity due to current-driven microinstabilities. This consideration provides a motivation for the form of \(\eta\) chosen here, which encompasses the location of strongest current. We define the nonideal region, \(D\), to be inside the surface defined by \(|j/\sigma| = 0.02(|j/\sigma|)_{\text{max}}\). The exact profile of \(\eta\), given by Eq. (12), has been chosen to ensure there is no return current within \(D\). This is expected to be the case if the rotational plasma flows at the tube footpoints possess a single sign in planes of constant \(z\). With the localization imposed on \(\eta\), the results presented here are qualitatively the same as in a model in which the return current is very diffuse. However, a similar analysis to that presented...
here may be carried out with \( \eta = \eta_0 = \text{const.} \) The results (given in the Appendix) also represent an example of flux tube disconnection with qualitatively similar properties to that outlined here.

Combining Eqs. (8) and (9) and taking \( \mathbf{E} = -\nabla \phi \) gives

\[
- \nabla \phi + \mathbf{v} \times \mathbf{B} = \frac{1}{\sigma} \mathbf{j}.
\]

(13)

The nonideal term \( \eta \mathbf{j} \) is now known, and we are left to deduce \( \phi \) and \( \mathbf{v} \). The component of Eq. (13) parallel to the magnetic field is given by

\[
- \nabla \phi \cdot \mathbf{B} = \frac{1}{\sigma} \mathbf{j} \cdot \mathbf{B},
\]

and so an expression for \( \phi \) can be found by integrating along the magnetic field lines, expressions for which are given by Eq. (10). Starting the integration from the initial condition \( \phi = \phi_0 (r_0, \theta_0) \) at \( z = 0 \), we deduce that

\[
\phi(r_0, \theta_0, s) = \int_0^s \frac{1}{\sigma} \mathbf{j} \cdot \mathbf{B} \, ds + \phi_0 (r_0, \theta_0).
\]

(14)

An equivalent expression for \( \phi \) in terms of \((r, \theta, z)\) can then be obtained using the inverse field line mappings given by Eq. (11).

The initial integration condition \( \phi_0 (r_0, \theta_0) \) is a free function that will affect the plasma velocity \( \mathbf{v} \), and so, for a solution confined to a finite region, it represents a boundary condition on the solution. We choose here to set \( \phi_0 (r_0, \theta_0) = 0 \) since this is the condition needed for a purely counter-rotational plasma velocity that is antisymmetric about \( z = 0 \). The corresponding potential, \( \phi \), is given as a function of \( r \) and \( z \) by

\[
\phi = -\frac{\sqrt{2\pi} \eta_0 k L}{\sigma_0 \mu a^3} (a^2 - r^2) e^{-5r^2/L^2} \text{erf} \left( \frac{\sqrt{2} z}{L} \right),
\]

(15)

where \( \text{erf}(\xi) \) is the error function defined by

\[
\text{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^\xi e^{-u^2} \, du.
\]

Thus, a jump in the electric potential across the nonideal region exists, with the maximum potential difference being along the central field line. In the nonideal region, the electric field is in the radial direction only, oppositely directed above and below the central plane \( (z = 0) \), and confined within the flux tube consisting of field lines that thread the nonideal region.

The component of the velocity perpendicular to the magnetic field can be deduced from Eq. (13) as

\[
\mathbf{v}_\perp = \frac{\mathbf{j} (r/\sigma) \times \mathbf{B}}{|\mathbf{B}|^2}.
\]

We use the freedom to add a component parallel to \( \mathbf{B} \) to ensure \( \nabla \cdot \mathbf{v} = 0 \),

\[
\mathbf{v} = \mathbf{v}_\perp - \frac{\mathbf{B}}{b_0}.
\]

The resultant velocity field is in the azimuthal direction only and given by

\[
\mathbf{v} = \frac{2 \eta_0 k r}{a} e^{-5r^2/L^2} \left[ \frac{\sqrt{2} \pi L}{a^3} (6a^2 - 5r^2) \text{erf} \left( \frac{\sqrt{2} z}{L} \right) + \frac{2e^{-2z^2/L^2}}{L^2} \theta \right],
\]

where \( \eta_0 = 1/\sigma_0 \mu \). An example is illustrated in Fig. 7. The

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**FIG. 7.** Plasma flows (a) above \((z=1)\) and (b) below \((z=-1)\) the central plane, \(z=0\). The flows are seen to be counter-rotational in nature. The solid line indicates the boundary 2% of the nonideal region in the same planes (parameters \( k=1 \), \( a=1 \), \( L=3 \), \( b_0=1 \), \( \mu=1 \), and \( \eta_0=1 \)).
flow is counter-rotational, i.e., rotates in an opposite sense above and below the \( z=0 \) plane (where it vanishes). The magnitude of the flow increases with distance from the central plane and becomes independent of height, \( z \), in the ideal region. Nonzero flow is confined to the flux tube consisting of the field lines threading the nonideal region.

How does this velocity compare with the order of magnitude estimate given by Eq. (5) in the preceding section? Above and below the nonideal region, we may obtain an estimate of the typical plasma velocity (which, in these regions, is independent of height) by finding the maximum value of \( v \) along a radial line. This maximum velocity turns out to be given by

\[
v_{\text{max}} = \frac{k \eta k L}{\alpha},
\]

where \( \kappa \) is the constant given by

\[
\kappa = \frac{9 + \sqrt{177}}{20} \sqrt{2 \pi (75 - 5 \sqrt{177})} \exp \left( \frac{\sqrt{177}}{4} - \frac{15}{4} \right)
\approx 5.327.
\]

The expression (16) is in qualitative agreement with the order of magnitude estimate given by Eq. (5). We have, therefore, confirmed the intuitive estimate with the analytical model.

The motivation for our choice of \( \eta \) is now apparent. Since the component of the current parallel to the magnetic field is given by

\[
j_1 = \frac{4 b_0 k}{\mu a^3} (a^2 - r^2) \exp \left( -\frac{r^2}{a^2} - \frac{z^2}{L^2} \right),
\]

there is a change in the sign of \( j_1 \) at \( r=a \). A uniform resistivity would then lead to flows in each \( r=\text{const} \) plane rotating in different senses for \( r<a \) and for \( r>a \). Although the magnitude of the rotational flow for \( r>a \) would be small, we choose to set it to zero for simplicity, by imposing a profile for \( \eta \) that ensures the nonideal region is contained within the surface \( r=a \). We have therefore obtained a kinematic model for the qualitative description of flux tube disconnection outlined in Sec. II.

In this reconnection process, all the field lines that thread through the nonideal region are continually changing their connections, with each field line reconnecting to others within the same surface \( r=r_0 \). This is one of the features that distinguishes the 3D case from the 2D case where field lines are cut and reconnected at a single point, and therefore the reconnection rate has a different interpretation in 3D. The field line that has the maximum difference in potential above and below the nonideal region is identified as the reconnection line. In the present model, for symmetry reasons, the reconnection line is the \( z \) axis. The reconnection rate is given by

\[
\frac{d\Phi_{\text{rec}}}{dt} = \int \mathbf{E}_d \mathbf{dl} = \int \mathbf{E} \cdot \mathbf{B} \mathbf{dl} = \int_{-\infty}^{\infty} \frac{4 \eta b_0 k}{a} e^{-\frac{2 \pi^2 a^2}{d}} dz
\]

or, after substituting for \( v_{\text{max}} \) from Eq. (16),

\[
\frac{d\Phi}{dt} = \frac{\sqrt{2 \pi} \eta b_0 k L}{a},
\]

where \( \kappa \) is the constant given by Eq. (17). Thus the reconnection rate in the kinematic model is, up to a constant factor, in agreement with the earlier qualitative estimate [Eq. (6)].

Previous analytical 3D reconnection models (see, for example, Ref. 29) have taken a uniform current and localized the nonideal region through localization of the resistivity \( \eta \) alone. Such models find that the reconnection rate is independent of the parameter controlling the radius of the nonideal region. Here we find that with a localized current, this parameter, \( a \), does become important in determining the reconnection rate.

**IV. POSSIBLE IMPLICATIONS OF THE MOMENTUM EQUATION**

Both the order of magnitude estimate (discussed in Sec. II) and the analytical model (discussed in Sec. III) for flux tube disconnection are kinematic analyses, i.e., they neglect the implications of the momentum equation. In this section we consider, from a qualitative perspective, the extent to which the momentum equation may effect the solutions. It was shown in Secs. II and III that some freedom exists in the solutions presented. Specifically, the flux tube may respond to an increase (decrease) in the magnitude of the rotational driving velocity \( v_0 \) either by an increase (decrease) in the length, \( l \), of the nonideal region or by an increase (decrease) in the number of turns present within the flux tube or by a combination of both effects. We wish to examine whether this freedom is inherent to the 3D process, or whether it arises through neglect of the momentum equation.

Assume, then, that the qualitative model presented in Sec. II also satisfies the momentum equation,

\[
\rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \mathbf{p} + \mathbf{j} \times \mathbf{B}.
\]

Consider the plasma pressure along a field line bounding the flux tube. In the central plane \( (z=0) \), the plasma velocity \( \mathbf{v} \) vanishes and so we may estimate the pressure at the edge of the tube as

\[
(\nabla \mathbf{p}) \mid_{\text{center}} + \mathbf{j} \times \mathbf{B} = 0 \Rightarrow |\nabla \mathbf{p}| \mid_{\text{center}} = j_c B_\theta = \frac{2 k^2 b_0^2}{\mu a}.
\]

In a plane of constant \( z \) above the nonideal region, the Lorentz force vanishes and
ideal region, that

\[ (\nabla p)_{\text{lop}} = \rho (v \cdot \nabla)v \Rightarrow |\nabla p|_{\text{lop}} = \frac{\rho v_0^2}{a}. \]  \tag{19} 

Now, since the pressure must be constant along a boundary field line, the estimates for \(|\nabla p|_{\text{center}}\) and \(|\nabla p|_{\text{lop}}\) given by Eqs. (18) and (19), respectively, must be equal, i.e., we have that

\[ v_0 = \sqrt{\frac{2}{\mu \rho}} k b_0, \]  \tag{20} 

or, rewriting in terms of the azimuthal Alfvén velocity, \(v_{A\theta} = k b_0/\sqrt{2 \mu \rho}\),

\[ v_0 = 2 v_{A\theta}. \]

We may now return to the estimate for the plasma velocity \(v_0\), given by Eq. (5) and obtained in the qualitative analysis of Sec. II, i.e.,

\[ v_0 = \frac{2 \eta k L}{a^2}. \]

Equating this last expression for \(v_0\) with that given by Eq. (20) determines the parameter \(L\), which determines the length of the nonideal region, in terms of the axial field strength and flux tube radius as

\[ L = \sqrt{\frac{1}{2 \mu \rho \eta}} b_0 a^2. \]  \tag{21} 

These estimates suggest that a change in the rotational driving velocity \((v_0)\) results in a change in the azimuthal magnetic field. The length \((L)\) of the nonideal region is determined by the axial field strength and tube radius. The expression for \(L\) given by Eq. (21) also allows us to estimate the ratio, \(L/a\), between the length and diameter of the nonideal region,

\[ \frac{L}{a} = \frac{a}{\eta} = v_A = R_m, \]

where \(v_A\) is the Alfvén velocity, \(v_A = b_0/\sqrt{2 \mu \rho}\), and \(R_m = v_A a / \eta\) is the magnetic Reynolds number based on the width of the flux tube. Thus the inclusion of the momentum equation in the qualitative analysis suggests that the length of the nonideal region is very much greater than its width, resulting in a long thin current sheet.

V. CONCLUSION

We have presented a stationary model for flux tube disconnection. The model considers a straight magnetic flux tube that has a localized twist present in its central region as a result of a counter-rotational driving velocity imposed at the footpoints of the magnetic flux tube. The model has a nonideal region that is localized in all three dimensions, and an electric field component parallel to the magnetic field is present within the nonideal region. These two properties are those required for the process to be considered as an example of global general magnetic reconnection.\(^{16}\) It differs in many respects from more traditional models of 2D and 3D reconnection; the magnetic field is not of X-type structure, and the field lines are continually cut throughout the diffusion region. The model provides a further demonstration that the characteristics of 3D reconnection are quite different from the 2D ones to which we have grown accustomed.

We expect flux tube disconnection to be a ubiquitous process in astrophysical plasmas. For example, the solar corona is permeated by a multitude of coronal loops, the ends of which are anchored in the photosphere where they are subject to photospheric motions. Spinning motions will tend to cause currents to build up in the overlying corona, allowing disconnection to occur, and allow for a heating of the plasma. In this paper, we have presented simple qualitative and quantitative models for the disconnection process so that its physical implications, in particular the rate of reconnection, may be determined.

An order of magnitude analysis, presented in Sec. II, allows us to understand, from a qualitative point of view, how the disconnection occurs. An increase in the rotational driving velocity of the footpoints results in an increase in the number of turns present within the twisted flux tube. The number of turns may be altered by increasing the strength of the poloidal field component, increasing the length of the nonideal region, or by a combination of both effects. A similar qualitative estimate of the reconnection rate has also been made, which was shown to be proportional to the rotational driving velocity and to the magnetic flux of the tube, i.e., the product of the magnetic field strength and the radius of the flux tube.

In Sec. III, we presented an analytical incompressible model of flux tube disconnection. Just as with several such 3D reconnection models, the analysis is kinematic, in that the effects of the equation of motion have been ignored. Instead, the implications of Ohm’s law and Faraday’s law have been considered. The analytical solutions confirm the estimates of the flux tube geometry and strength and reconnection rate in relation to the footpoint velocity. A qualitative estimate from the equation of motion in Sec. IV implied that the ratio \((L/a)\) of the diffusion region length to width is of the order of magnetic of the Reynolds number. In turn this implies that normally the reconnection is slow and can only be fast when \(L/H = R_m\).

The kinematic model offers interesting insights into the 3D reconnection process, and in particular into how the flux tube parameters may be determined. In future it would be useful also to include (either analytically or computationally) the equation of motion in the analysis. Such an inclusion may help to constrain some of the relations between parameters (such as \(L\) and \(H\) that are free in this model, and would enable several other important questions to be addressed. For example, plasma acceleration is thought to be an important feature of reconnection. This is suggested by the equation of motion since regions of strong current may well also have strong Lorentz forces, as is the case in the 2D models. In turn the Lorentz forces would give rise to plasma acceleration unless they happen to be balanced exactly by the remaining forces acting on the fluid. Another argument suggesting plasma acceleration may take place is that during reconnection, magnetic energy is converted to other forms, and so kinetic energy may be one of these. Thus in the future when
the equation of motion is included, we shall be able to determine how the plasma is accelerated in the model. Another important question regards the stability of the solution presented. For a stability analysis, either knowledge of the full exact solution will be required or the problem will need to be approached numerically. A computational model of the process is therefore an important future step and may also allow questions regarding the dynamic accessibility of the solution to be addressed.

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APPENDIX: SOLUTION WITH UNIFORM DIFFUSIVITY

In Sec. III, we presented a model for flux tube disconnection in which the magnetic diffusivity $\eta$ is localized. If instead we take a uniform diffusivity $\eta$, then the solutions for $\phi$ and $v$ are given instead by

$$\phi = -\frac{2\sqrt{\pi \beta k L}}{\alpha_0 \mu a^3}(a^2 - r^2)e^{-\gamma_0 a^2} \text{erf}\left(\frac{z}{L}\right),$$

$$v = \frac{4\eta kr}{a} e^{-\gamma_0 a^2} \left[ \frac{\sqrt{\pi} L}{4a}(2a^2 - r^2)\text{erf}\left(\frac{z}{L}\right) + \frac{z}{L^2}e^{-\gamma_0 a^2} \right] \theta.$$

The solution is seen to be similar to that given in Sec. III and its interpretation is still of flux tube disconnection. In this solution, the outermost section of the flux tube undergoes a rotating motion that is opposite to that of the innermost section.