On the nature of chaotic regions in dissipative hydrodynamics and magnetohydrodynamics

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A region with chaotic magnetic field lines where the magnetic field ($\mathbf{B}$) and plasma velocity ($\mathbf{v}$) are continuous and differentiable and satisfy the dissipative incompressible magnetohydrodynamic equations with magnetic diffusivity η and kinematic viscosity ν is considered. It is proved then that if $\mathbf{v}\times\mathbf{B}$ and $(\nabla\times\mathbf{v})\times\mathbf{v}$ are potential, the structurally stable solutions describing such chaotic regions are characterized by a decaying linear magnetic force-free field and Beltrami flow of the form $\mathbf{B}=B_0\exp(-\alpha^2\eta t)\mathbf{b}$, $\mathbf{v}=v_0\exp(-\alpha^2\nu t)\mathbf{b}$, where $\mathbf{b}=\mathbf{b}(\mathbf{r})$ such that $\nabla\times\mathbf{b}=\alpha\mathbf{b}$, $\nabla\cdot\mathbf{b}=0$ and B_0, v_0, and α are constants. Purely hydrodynamic flows are a particular case with $B_0=0$. A simple example of a chaotic force-free field is also constructed. [S1070-664X(99)03804-5]

Magnetostatic equilibria and analagous Euler flows may in general consist of laminar domains enclosing ergodic regions,[1] both in astrophysical and laboratory plasmas. Chaos has, for example, been studied in connection with the dynamo generation of astrophysical magnetic fields,[2] especially the effect of chaotic ABC flows on magnetic fields.[3] Also, in the fusion literature it has long been recognized that magnetic surfaces may be destroyed by nonaxisymmetric perturbations to produce regions filled with chaotic field lines.[4–6] Chaotic fields have been considered in, for example, tokamaks[7,8] and reversed-field pinches.[9,10]

A natural question that we are raising here is therefore: what types of solution to the resistive magnetohydrodynamic (MHD) equations are possible when the magnetic field is chaotic? In Sec. II we introduce the basic equations and in Sec. III we prove the main result about a particular class of admissible solutions. Then finally in Sec. IV we present a simple analytical example of a chaotic field that, unlike the ABC field, is not periodic.

The magnetohydrodynamic (MHD) equations for an incompressible medium with uniform density ρ, resistivity $1/\sigma$, kinematic viscosity ν, and magnetic permeability μ in a gravitational field can be written as

$$\partial\mathbf{v}/\partial t+\boldsymbol{\omega}\times\mathbf{v}=-\nabla P+\mathbf{j}\times\mathbf{B}/\rho+\nu\nabla^2\mathbf{v}, \tag{1}$$

$$-\partial\mathbf{A}/\partial t-\nabla\phi+\mathbf{v}\times\mathbf{B}=\mathbf{j}/\sigma, \tag{2}$$

where

$$P=p/\rho+v^2/2+G, \tag{3}$$

$$\boldsymbol{\omega}=\nabla\times\mathbf{v}, \tag{4}$$

$$\nabla\cdot\mathbf{v}=0, \tag{5}$$

$$\mathbf{j}=\nabla\times\mathbf{B}/\mu, \tag{6}$$

$$\nabla\cdot\mathbf{B}=0. \tag{7}$$

We assume that all the physical parameters (velocity $\mathbf{v}$, vorticity $\boldsymbol{\omega}$, pressure p, gravitational potential G, magnetic field $\mathbf{B}$, current density $\mathbf{j}$, vector potential $\mathbf{A}$, and electric potential ϕ) are continuous and differentiable functions of the three-dimensional spatial variable $\mathbf{r}$ and time t. This assumption is trivial for laminar regions, but it leads to profound consequences for chaotic regions. To show this, note first that the scalar product of $\mathbf{B}$ and Eq. (2) yields

$$\mathbf{B}\cdot\nabla\phi=-\mathbf{B}\cdot(\mathbf{j}/\sigma+\partial\mathbf{A}/\partial t), \tag{8}$$

which can be considered as an equation for ϕ if the evolution of the magnetic field is given, so that the vector potential $\mathbf{A}$ at each moment is defined up to the gradient of some gauge g by $\nabla\times\mathbf{A}=\mathbf{B}$. It is always possible to absorb g into ϕ, so that Eq. (8) can be considered as a gauge equation for ϕ in MHD.[11]

The magnetic field lines here play the role of characteristics and the general solution of Eq. (8) is

$$\phi=\phi_0-\int_0^s(\mathbf{B}/B)\cdot(\mathbf{j}/\sigma+\partial\mathbf{A}/\partial t)ds, \tag{9}$$

which defines the value of ϕ at each point $\mathbf{r}(s)$ that is mapped according to $d\mathbf{r}/ds=\mathbf{B}/B$ along a field line over a distance s from a starting point $\mathbf{r}_0$, where $\phi(\mathbf{r}_0)\equiv\phi_0$, say. If ϕ is continuous at $\mathbf{r}_0$, the integral in (9) must tend to zero as $\mathbf{r}(s)$ approaches $\mathbf{r}_0$. In a laminar region the latter would occur if the integral is taken along a closed magnetic field line. Then it would imply that, if the circulation of $\partial\mathbf{A}/\partial t$ around such a line does not vanish, it must be compensated

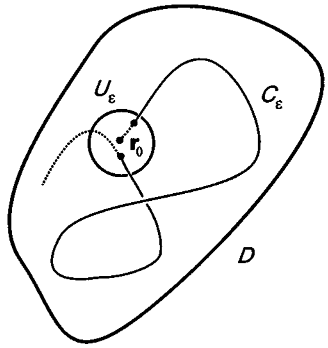

FIG. 1. ''Almost closed'' paths $\mathcal{C}_\epsilon$ formed by a magnetic field line in an ergodic region $\mathcal{D}$ outside a neighborhood $\mathcal{U}_\epsilon$ of a point $\mathbf{r}_0$ belonging to $\mathcal{D}$.

by the corresponding circulation of the current density $\mathbf{j}$. In the limit of large σ this implies generally the development of a strong current along the closed field line, which is a topological reason for the magnetic reconnection process. Here it is important to note that such closed field lines in asymmetric laminar regions are isolated one-dimensional objects and so they cannot lead to a volume constraint on the evolution of the magnetic field.

Let $\mathbf{r}_0$ be now located on the field line which ergodically fills some region $\mathcal{D}$ in the volume and $\mathcal{U}_\epsilon$ be its neighborhood of size ϵ (Fig. 1). Then as s increases in Eq. (9) this field line will leave $\mathcal{U}_\epsilon$ and eventually come back to $\mathcal{U}_\epsilon$ many times, thereby forming ''almost closed'' paths $\mathcal{C}_\epsilon$ outside $\mathcal{U}_\epsilon$.

Note, first, that the continuity of ϕ requires from Eq. (9) that the line integral of the field $\mathbf{j}/\sigma+\partial\mathbf{A}/\partial t$ over each $\mathcal{C}_\epsilon$ must vanish as ϵ tends to zero. Second, no matter how small ϵ is, the paths $\mathcal{C}_\epsilon$ fill the region $\mathcal{D}/\mathcal{U}_\epsilon$ in the sense that they come arbitrarily close to any point. Third, any even very small perturbation or fluctuation of $\mathbf{B}$, which is unavoidable in reality, would cause in the generic case an enormous change of the path $\mathcal{C}_\epsilon$ with a corresponding change in ϕ if the integral in Eq. (9) is not trivial. This implies that the path $\mathcal{C}_\epsilon$ may be considered as arbitrary for some class of structurally stable solutions.

These three statements motivate us to consider a form of the above field, belonging to such a class, that is purely potential in $\mathcal{D}$, i.e.,

$$\mathbf{j}/\sigma+\partial\mathbf{A}/\partial t=\nabla f. \tag{10}$$

Then we find from Eq. (2) two consequences; first,

$$\partial\mathbf{B}/\partial t=\eta\nabla^2\mathbf{B}, \tag{11}$$

where $\eta=1/(\sigma\mu)$ is magnetic diffusivity, and, second,

$$\mathbf{v}\times\mathbf{B}=\nabla(\phi+f).$$

The latter in turn implies $\mathbf{B}\cdot\boldsymbol{\nabla}(\phi+f)=0$, which due to the ergodic behavior of $\mathbf{B}$ yields $\phi+f=F(t)$ and so $\mathbf{v}\times\mathbf{B}=0$ or $\mathbf{v}=k(\mathbf{r},t)\mathbf{B}$. By substituting this into Eq. (5) and taking into account Eq. (7) we obtain now that $\mathbf{B}\cdot\nabla k=0$. This yields again $k\equiv k(t)$ and hence

$$\mathbf{v}=k(t)\mathbf{B}, \tag{12}$$

i.e., the flow is also chaotic in the region being considered and it has a streamline coinciding with the ergodic magnetic field line.

The scalar product of $\mathbf{v}$ with Eq. (1) and use of Eq. (12) yields

$$\mathbf{v}\cdot\nabla P=-\mathbf{v}\cdot(\partial\mathbf{v}/\partial t-\nu\nabla^2\mathbf{v}) \tag{13}$$

which is similar to Eq. (8). Assuming now continuity of P and the ergodic property of $\mathbf{v}$, we consider as before velocities that satisfy $\partial\mathbf{v}/\partial t-\nu\nabla^2\mathbf{v}=\nabla h$ so as to be able to find a particular class of solutions compatible with our assumptions. So we have again two consequences first,

$$\partial\boldsymbol{\omega}/\partial t=\nu\nabla^2\boldsymbol{\omega} \tag{14}$$

and after using Eq. (12), secondly,

$$[1/(k^2\mu\rho)-1]\boldsymbol{\omega}\times\mathbf{v}=\nabla(P+h).$$

Let $k\neq 1/\sqrt{\mu\rho}$; then it follows that $\mathbf{v}\cdot\boldsymbol{\nabla}(P+h)=0$, which yields $P+h=H(t)$ and so $\boldsymbol{\omega}\times\mathbf{v}=0$ or $\boldsymbol{\omega}=\alpha(\mathbf{r},t)\mathbf{v}$. The divergence of the latter together with Eq. (5) gives $\mathbf{v}\cdot\boldsymbol{\nabla}\alpha=0$, which means for an ergodic $\mathbf{v}$ that $\alpha\equiv\alpha(t)$. Using now these conclusions and the relationship $\boldsymbol{\nabla}^2\boldsymbol{\omega}=-\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\omega})$, one can easily derive from Eq. (14) and its curl that $\alpha\equiv$const, while $\mathbf{v}$ must satisfy

$$\partial\mathbf{v}/\partial t=-\alpha^2\nu\mathbf{v}. \tag{15}$$

So from Eqs. (6) and (12) we have $\mathbf{j}=\alpha\mathbf{B}/\mu$, which in combination with Eqs. (6) and (11) yields

$$\partial\mathbf{B}/\partial t=-\alpha^2\eta\mathbf{B}. \tag{16}$$

Thus, the assumptions of ergodicity of magnetic field lines, differentiability of physical values in chaotic regions and $\mathbf{v}\times\mathbf{B}$ and $\boldsymbol{\omega}\times\mathbf{B}$ being potential imply that the solutions of Eqs. (1)–(7) are of the form

$$\mathbf{v}=v_0\exp(-\alpha^2\nu t)\mathbf{b}, \tag{17}$$

$$\mathbf{B}=B_0\exp(-\alpha^2\eta t)\mathbf{b}, \tag{18}$$

where v_0 and B_0 are the corresponding scales, while $\mathbf{b}\equiv\mathbf{b}(\mathbf{r})$ is a dimensionless Beltrami field satisfying

$$\boldsymbol{\nabla}\times\mathbf{b}=\alpha\mathbf{b},\quad \boldsymbol{\nabla}\cdot\mathbf{b}=0,\quad \alpha=\text{const}. \tag{19}$$

Direct substitution of Eqs. (17)–(18) into Eqs. (1)–(7) shows that they are satisfied identically if Eq. (19) is fulfilled and $p/\rho+v^2/2+G=P\equiv$const. Note also that Eqs. (17)–(18) represent, of course, exact solutions even if the Beltrami field $\mathbf{b}$ is not chaotic. Purely hydrodynamic flows which are ergodic in some region $\mathcal{D}$ simply represent a particular case of magnetohydrodynamic flows with $B_0=0$ in (18). The part of the above proof starting from Eq. (13) refers to this case if one puts formally $k^{-1}=0$.

The special case of $k\equiv 1/\sqrt{\mu\rho}$ is not generic, since it requires that $\nu\equiv\eta$ for nontrivial solutions. This case corresponds to an Alfvén wave, whose velocity and magnetic field related by $\mathbf{v}=\mathbf{B}/\sqrt{\mu\rho}$ decay by resistive and viscous diffusion.

The reasonableness of our basic result may be seen as follows. Write Ohm's law directly in terms of the electric field so that $\mathbf{E}+\mathbf{v}\times\mathbf{B}=\mathbf{j}/\sigma$ whose scalar product is $\mathbf{B}\cdot(\mathbf{E}$

$-\mathbf{j}/\sigma)=0$. Integrating this from some initial point $\mathbf{r}(0)$ along a field line $\mathbf{r}(s)$ defined by $d\mathbf{r}/ds=\mathbf{B}/B$ gives $\int_0^s \mathbf{B}\cdot(\mathbf{E}-\mathbf{j}/\sigma)/B\,ds$. If the field line is chaotic and ergodically fills a volume, the result that the above line integral must vanish for all values of s clearly places very strong conditions on the vector field $\mathbf{E}-\mathbf{j}/\sigma$. There may well be solutions with $\mathbf{E}-\mathbf{j}/\sigma$ nonzero, but the most obvious solution is to set $\mathbf{E}=\mathbf{j}/\sigma$ throughout the volume, so that $\mathbf{v}\times\mathbf{B}=\mathbf{0}$ and similarly the scalar product with $\mathbf{B}$ of the equation of motion leads to $\mathbf{j}\times\mathbf{B}=\mathbf{0}$, as before.

The purpose of this section is to construct a simple example of a force-free field that exhibits chaotic behavior. Solutions in spherical polar coordinates (r,θ,ϕ) to the linear force-free magnetic field Eq. (3.10) may be expressed in the series form

$$\mathbf{b}=\mathbf{r}\times\nabla\Psi+\frac{1}{\alpha}\nabla\times(\mathbf{r}\times\nabla\Psi), \tag{20}$$

where

$$\Psi=\sum_{m=0}^{\infty}\sum_{n\leqslant m}^{\infty} j_m(\alpha r)P_m^n(\cos\theta)e^{in\phi}. \tag{21}$$

Here j_m are the spherical Bessel functions

$$j_m(x)=\sqrt{\pi/2x}\,J_{m+(1/2)}(x),$$

and $J_{m+(1/2)}$ are ordinary Bessel functions, $P_m^n(\cos\theta)$ are the associated Legendre functions and

$$P_m^n(x)=\frac{(1-x^2)^{n/2}}{2^m m!}\frac{d^{(n+m)}}{dx^{(n+m)}}(x^2-1)^m.$$

Individual components of Eq. (20) arising from the P_m^0 terms have rotational symmetry about the $\theta=0,\pi$ axis, and may be written in the semi-Hamiltonian form

$$\mathbf{b}_m^0=\nabla\phi\times\nabla H_m(r,\theta)+\alpha^{-1}H_m(r,\theta)\nabla\phi, \tag{22}$$

where

$$H_m(r,\theta)\equiv r\sin\theta\frac{\partial}{\partial\theta}(j_m(\alpha r)P_m(\cos\theta))$$

is constant along the field lines of $\mathbf{b}_m^0$, since $\mathbf{b}_m^0\cdot\nabla H_m=0$.

Field lines of the axisymmetric fields $\mathbf{b}_m^0$ form sets of nested surfaces around the axis of symmetry, each surface being parameterised by a constant value of H_m. The *winding number* of a given field line may be defined as the ratio of the periods of the variations in r and ϕ coordinates along the field line (or equivalently for the θ and ϕ coordinates). Field-lines with irrational winding number will eventually fill the surface of constant H_m on which they lie. Field-lines with rational winding number are closed; corresponding rational surfaces of constant H_m are formed from an infinite number of such closed field lines.

When a small symmetry-breaking perturbation is added to a field $\mathbf{b}_m^0$, the field lines will no longer be constrained to lie on surfaces of constant H_m. However, the KAM (Kolmogorov–Arnold–Moser) theorem predicts that many of the nested torilike surfaces do survive; surfaces close to the center of the tori structure are perturbed but remain as nested surfaces. Beyond the region where the KAM surfaces retain their integrity the structure is likely to break down, losing structure in a descent to stochasticity.

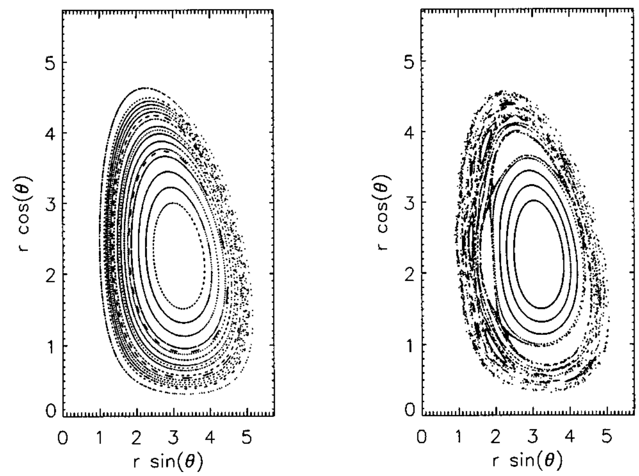

FIG. 2. Poincare sections through the plane $\phi=1.5$ for perturbed field (4.5) at distinct values of ϵ.

However, between the region of stochasticity and the KAM tori, other interesting features are likely to arise. On rational surfaces some of the closed field lines may survive the perturbation, while the field lines nearby tend to form new torilike surfaces around them. On a Poincare section plot these features appear as characteristic islandlike structures. The region between these new tori and the original tori may well be occupied by chaotically wandering field lines that ergodically fill some region between the sets of the nested tori.

To construct a simple example of a force-free field with chaotic behavior, we modify a cylindrically symmetric field $\mathbf{b}_2^0$ by adding a force-free perturbation of the following simple (Beltrami ABC-field) type

$$\mathbf{b}_{ABC}=\sin(\alpha y+1)\mathbf{i}-\cos(\alpha y+1)\mathbf{k}. \tag{23}$$

Figure 2 shows a Poincare plot through the plane $\phi=1.5$ for the field

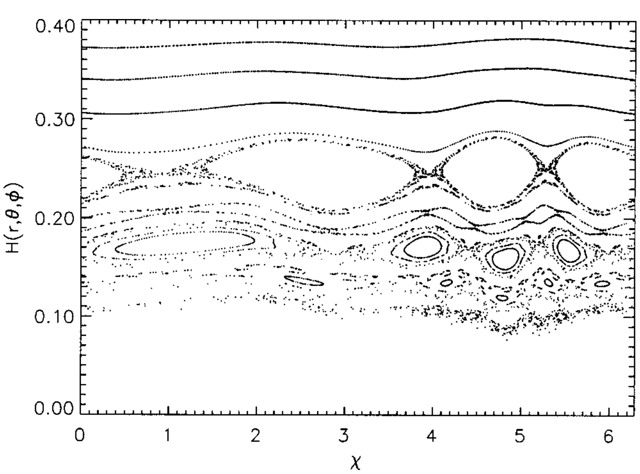

FIG. 3. Poincare section plotted in adapted coordinates.

$$\mathbf{b} = \mathbf{b}_2^0 + \epsilon \mathbf{b}_{ABC} \tag{24}$$

with $\alpha = 1$ for two values of ϵ which suggest the onset of chaotic behavior as the size of the perturbation is increased.

Figure 3 shows the same data points as in the second Poincare section from Fig. 2, plotted in adapted coordinates that highlight the standard nature of the effect of symmetry breaking. H is the function defining the invariant surface of the unperturbed field $\mathbf{b}_2^0$. H decreases from the center of the unperturbed tori to zero at the first sphere at which the b_r and b_ϕ components of $\mathbf{b}_2^0$ vanish. The coordinate χ is the angle measured around the center of the unperturbed tori, with $\chi = 0$ in the upward vertical direction.

In Fig. 3 the invariant tori are located near the center of the structure at high H. As H decreases, islandlike structures appear near certain rational surfaces, surrounded by regions of chaos. As H decreases further the structure breaks down through a region of chaos, beyond which the field lines are no longer bounded.

This paper has addressed the question: what constraint is imposed on the dissipative incompressible MHD flows by the assumption of continuity and differentiability of all the physical values in a region with chaotic behavior of magnetic field lines. These assumptions imply that one of the classes of structurally stable solutions describing such chaotic regions is characterized by a decaying linear magnetic force-free field and Beltrami flow of the form given by Eqs. (17)–(19). An example of such a field containing chaotic regions is considered in Sec. IV. One can also obtain purely hydrodynamic solutions from these MHD solutions by setting the magnetic field to zero. This wide class of solutions does not exclude the appearance in MHD flows of chaotic regions with some other spatial and temporal behavior of the magnetic and velocity fields and, especially, with anomalous transport in such devices as tokamaks. However, the obtained solutions demonstrate that the condition of smoothness of the values imposes significant restrictions on the properties of the chaos.

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[1] H. K. Moffatt, J. Fluid Mech. **159**, 359 (1985).
[2] M. R. E. Proctor and A. D. Gilbert, *Lectures on Solar and Planetary Dynamos* (Cambridge University Press, Cambridge, 1994).
[3] A. D. Gilbert, Proc. R. Soc. London, Ser. A **339**, 627 (1992).
[4] M. N. Rosenbluth, R. Z. Sagdeev, J. B. Taylor, and G. M. Zaslavski, Nucl. Fusion **6**, 297 (1966).
[5] A. H. Boozer, Phys. Fluids **26**, 1288 (1983).
[6] J. M. Finn and T. M. Antonsen, Phys. Fluids **26**, 3540 (1983).
[7] J. M. Finn, Nucl. Fusion **15**, 845 (1975).
[8] J. Wesson, *Tokamaks* (Oxford University Press, Oxford, 1997).
[9] D. D. Schnack, E. J. Caramana, and R. A. Nebel, Phys. Fluids **28**, 321 (1985).
[10] J. M. Finn, Phys. Fluids B **4**, 1262 (1992).
[11] M. D. Kruskal and R. M. Kulsrud, Phys. Fluids **1**, 265 (1958).