



## LINEAR THEORY OF STEADY X-POINT MAGNETIC RECONNECTION

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### ABSTRACT

Slow magnetic reconnection at a neutral X-point of a two-dimensional magnetic field is studied in an incompressible viscous resistive fluid. It is shown analytically that the combined effect of viscosity and resistivity resolves the current singularity appearing in both the ideal and resistive magnetohydrodynamic approximations at the X-point and along the separatrices when the flow is allowed to cross them. A previous attempt had retained a weak singularity at third order. A two-parameter family of exact solutions describing the structure of the flow and current density distribution is found for the corresponding basic equations.

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### INTRODUCTION

The problem considered in this paper has an interesting prehistory. It started when Syrovatskii (1979) analyzed slow or inertia-less steady flows of plasma in the vicinity of an X-point, using the so-called "strong magnetic field approximation" for ideal magnetohydrodynamics (MHD). In particular, he found that such flows are necessarily singular at the separatrices.

Recently, Craig and Rickard (1994) have shown in a similar (but resistive and incompressible) approximation that one can find a steady regular solution for the fluid motion near the X-point configuration if one supposes an absence of flow across the separatrices. Then it was proved by Priest *et al.* (1994) that the absence of trans-separatrix flow is an inherent property of inertia-less resistive inviscid flows. Can magnetic reconnection with nonvanishing slow flow across the separatrices therefore be obtained when both resistivity and viscosity are included? The first attempt in this direction (Priest *et al.*, 1994) was only partially successful, since the resulting expression for the stream function has a discontinuity in the third derivative across the separatrices, which is not physically acceptable. Below we show how this discontinuity can actually be resolved by the combined effect of resistivity and viscosity.

### BASIC EQUATIONS

We shall study slow MHD flows of a resistive and viscous incompressible fluid in an X-type magnetic configuration. Suppose that these flows are slow enough that their influence on the magnetic field is a small inertia-less perturbation of some unperturbed magnetostatic configuration. As the latter unperturbed state we adopt a potential magnetic field in the neighbourhood of a two-dimensional null point, namely  $\mathbf{B}_0 \equiv (B_{0x}, B_{0y}) = (-x, y)$  in a system of coordinates for which the axes coincide with the separatrix field lines. These  $(x, y)$ -coordinates are more suitable for studying the problem than the conventional  $(x', y')$ -coordinates (Figure 1), and so only the final results will be presented in  $(x', y')$ -coordinates.

The incompressible flow can be described in terms of a stream function ( $\psi$ ) that satisfies the following linear dimensionless equation (see the details in (Priest *et al.*, 1994))

$$(\mathbf{B}_0 \cdot \nabla)^2 \psi - \epsilon \nabla^2 \nabla^2 \psi = 0. \quad (1)$$

The dimensionless parameter  $\epsilon = 1/(Re R_{me}) \equiv \nu\eta/(L_e v_{Ae})^2$  is defined in terms of the ordinary ( $Re = L_e v_{Ae}/\nu$ ) and magnetic ( $R_{me} = L_e v_{Ae}/\eta$ ) Reynolds numbers, both based on the Alfvén speed  $v_{Ae} = B_e/(\mu\rho)^{1/2}$ . As scales for the corresponding values we use the "external" flow speed  $v_e$  and magnetic field  $B_e$  at a global distance  $L_e$  from the X-point located in a uniform incompressible fluid with density  $\rho$ , magnetic permeability  $\mu$ , resistivity  $\eta$  and kinematic

viscosity  $\nu$ . Eq. (1) is the  $z$ -component of the curl of an equation of motion in which the inertia term has been neglected due to the assumed slowness of the flow. For much the same reason the perturbed magnetic field is also negligible in comparison with  $\mathbf{B}_0$ , so that the Lorentz force ( $\simeq j\hat{z} \times \mathbf{B}_0$ ) is due to the interaction of the potential field  $\mathbf{B}_0$  with the perturbed current density  $j$ . The latter is determined by Ohm's law

$$j/R_{me} = 1 + \mathbf{B}_0 \cdot \nabla\psi, \quad (2)$$

where the perturbed magnetic field is again neglected and, due to the assumed incompressibility of the flow, the velocity  $\mathbf{v} = \nabla \times (\psi\hat{z})$ . So the curl of the Lorentz force gives the first term of Eq. (1), while the curl of the viscous force  $\nu\nabla^2\mathbf{v}$  gives the second term.

If the viscosity is negligible so that  $\epsilon \ll 1$ , then Eq. (1) becomes simply  $(\mathbf{B}_0 \cdot \nabla)^2\psi = 0$ . After one integration along  $\mathbf{B}_0$  this leads to Eq. (2) in which, however, the current density  $j$  is now not arbitrary as before but is constant along field lines, i.e.  $j \equiv j(A_0)$ , where  $A_0 = -xy$  is the unperturbed flux function corresponding to  $\mathbf{B}_0$ . A subsequent integration of Eq. (2) along  $\mathbf{B}_0$  with the condition of flow symmetry (requiring that  $\psi = 0$  at  $x = y$ ) yields the resistive solution  $\psi = \frac{1}{2}(1 - R_{me}^{-1}j(A_0))\log|x/y|$ . In the limit of small resistivity, i.e.  $R_{me} \rightarrow \infty$ , this reduces to the ideal solution  $\psi = \frac{1}{2}\log|x/y|$  such that both of the velocity components  $v_x = -1/y$  and  $v_y = -1/x$  become singular at the separatrices  $y = 0$  and  $x = 0$ , respectively. At first sight, it may be thought that these singularities can be resolved in the inviscid resistive expression by a suitable choice of the current distribution  $j(A_0)$ . This possibility has been explored by Craig and Rickard (1994), who considered electric currents that are highly concentrated near the separatrices. However, in this case there is only a simple magnetic diffusion and no advection of magnetic flux across the separatrices – a result that has been derived in a general form by Priest *et al.* (1994) as the so-called Anti-Reconnection Theorem. So let us turn now to the combined effect of viscosity and resistivity described by Eq. (1). It should be noted first that the parameter  $\epsilon$  in Eq. (1) may be absorbed by a simple change of variable such that  $x = \epsilon^{1/4}\tilde{x}$ ,  $y = \epsilon^{1/4}\tilde{y}$ . Applying this transformation to Eq. (1) and then omitting the tildes, we obtain

$$\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)^2\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2\psi, \quad (3)$$

which will now be considered as our basic equation; its solutions give those of Eq. (1) by a simple rescaling of variables.

### EXACT SOLUTION PARTIALLY RESOLVING THE SEPARATRIX SINGULARITY

The additively separable structure of Eq. (3) prompts us to search for a particular solution in the form:  $\psi = f(x) - g(y)$ . Substituting this expression into Eq. (3), we find that the function  $f(x)$  must satisfy

$$x^2\frac{d^2f}{dx^2} + x\frac{df}{dx} = \frac{d^4f}{dx^4}. \quad (4)$$

A similar equation is obtained for the function  $g(y)$ , so the assumed form of solutions is compatible with our basic Eq. (3). Such solutions describe a linear superposition of two unidirectional fluid flows parallel to the separatrices. Generally such flows are different, since the functions  $f(x)$  and  $g(y)$  are not necessarily identical. However, we shall here assume them to be the same to have the simplest symmetrical flows.

Eq. (4) has appeared already (in Priest *et al.*, 1994) as an equation describing an *approximate* self-similar flow at the separatrix layer. One can see now, however, that its significance is surprisingly much wider – in fact it describes a family of *exact* solutions of Eq. (3). In spite of the differences in meaning and in independent variables, we can use all the results obtained by Priest *et al.* (1994) about the properties of the solutions of Eq. (4). First of all, there is a single solution  $F(x)$  of Eq. (4) which has the desired logarithmic asymptotics, i.e.  $F(x) \rightarrow 1/2\log|x|$  as  $x \rightarrow \infty$ . It may be written explicitly in terms of modified Bessel functions  $I_{\pm\frac{1}{4}}(\xi^2/4) \equiv \mathcal{I}_{\pm}(\xi)$  and  $K_{\frac{1}{4}}(\xi^2/4) \equiv \mathcal{K}(\xi)$  of order one quarter, so that the required solution is

$$\psi = F(x) - F(y), \quad \text{where} \quad F(x) = \frac{1}{4}\int_0^x \xi \mathcal{I}_+(\xi) \mathcal{K}(\xi) d\xi. \quad (5)$$

When substituted into Eq. (2) it yields a current density

$$j/R_{me} = J(x) + J(y), \quad \text{where} \quad J(x) = \frac{1}{2} - \frac{x^2}{4}\mathcal{I}_+(x)\mathcal{K}(x). \quad (6)$$

This solution demonstrates that the logarithmic singularity of the ideal MHD flow at the separatrix  $x = 0$  (or  $y = 0$ ) is indeed resolved by the combined effect of resistivity and viscosity, which is manifested in the appearance

of separatrix current spikes (Figure 2). The results are qualitatively the same as in (Priest *et al.*, 1994), whose approximate solution behaves in a similar way. Moreover, near the separatrices and far from the origin both solutions become asymptotically identical, so the above exact solution (5) is a *limiting* one for that approximate solution.

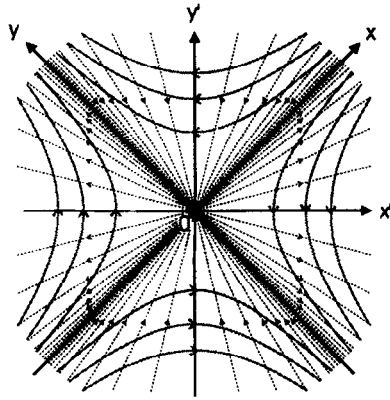


Fig. 1. The convenient  $(x, y)$  and conventional  $(x', y')$  systems of coordinates used for studying MHD flow near an X-point of unperturbed magnetic configuration whose field lines are shown by thick grey curves, while thin dashed lines represent the stream lines of an ideal MHD flow perturbing such a configuration.

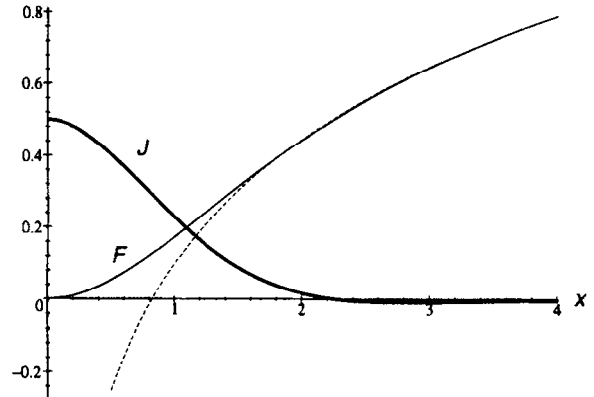


Fig. 2. The profiles of the stream function  $F(x)$  (thin solid curve) and corresponding current density  $J(x)$  (thick solid curve) describing a visco-resistive unidirectional shear flow with partially resolved logarithmic singularity at  $x = 0$  of an ideal MHD flow. The asymptotic behaviour  $(1/2 \log x + 0.097)$  of  $F(x)$  is shown by the dashed curve.

Even though the solution (5) is exact, it has the same disadvantage as the approximate solution of having a discontinuous third derivative across the separatrices. This is because both of them are constructed on the basis of the function  $F(x)$  (5) which has a jump in  $F'''$  at  $x = 0$ . Such a jump implies a discontinuity in the tangential component of the viscous force  $\rho\nu\nabla^2\mathbf{v}$  at the separatrices, which in turn means a corresponding discontinuity of pressure there. So the above visco-resistive solution (5) resolves the singularity in the ideal MHD flow only partially, i.e. to within this weak discontinuity.

PHYSICALLY ACCEPTABLE FAMILY OF PIECE-WISE ANALYTICAL SOLUTIONS

A further analysis shows that the desired smoothness (of viscous force and pressure) and asymptotic behaviour can be obtained with a solution having a more general form, namely

$$\psi = f_0(x) + f_1(x)\frac{y^2}{2} - f_0(y) - f_1(y)\frac{x^2}{2}, \tag{7}$$

where the unknown functions  $f_0$  and  $f_1$  represent, respectively, the unidirectional and non-unidirectional parts of the flow connected with one another by the following system of equations:

$$x^2\frac{d^2 f_0}{dx^2} + x\frac{df_0}{dx} = \frac{d^4 f_0}{dx^4} + 2\frac{d^2 f_1}{dx^2}, \quad x^2\frac{d^2 f_1}{dx^2} - 3x\frac{df_1}{dx} + 4f_1 = \frac{d^4 f_1}{dx^4}. \tag{8}$$

These equations can be solved analytically in a rather nontrivial way that will be described in an extended version of this paper, while here we present only the resulting expressions

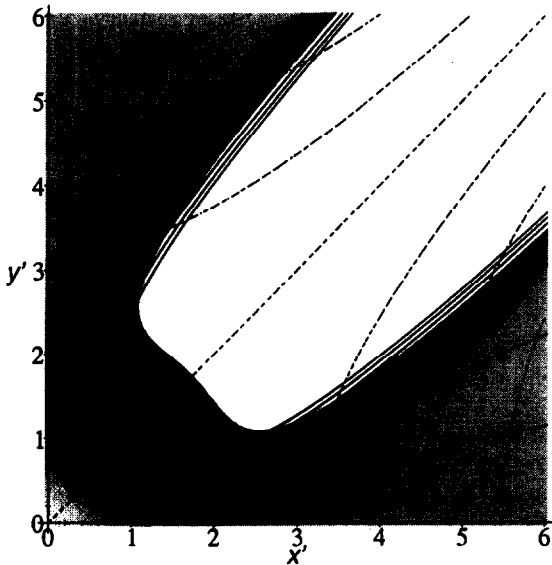
$$f_0(x) = 2 \int_{-\infty}^{+\infty} \left( \int_{\xi}^{\infty} \tilde{h}(\xi') d\xi' \right) G(\xi, x) d\xi, \quad f_1(x) = \frac{1}{2} \int_x^{\infty} (x - \xi)^2 \tilde{h}(\xi) d\xi. \tag{9}$$

They contain a piece-wise smooth function  $\tilde{h} = \xi (c_1^i \mathcal{I}_+(\xi)^2 + c_2^i \mathcal{K}(\xi)^2 + c_3^i \mathcal{I}_+(\xi) \mathcal{K}(\xi))$ , where  $c_j^i$  are constants determined in a special way such that  $i = 1, 2$  or  $3$  if  $|\xi|$  belongs, respectively, to the intervals  $[0, x_1], (x_1, x_2]$  or  $(x_2, \infty)$ ; the positive values  $x_1$  and  $x_2$  are arbitrary parameters. The function  $G(\xi, x)$  entering into Eq. (9) is a Green's function of the first equation in the system (8) and can be written as

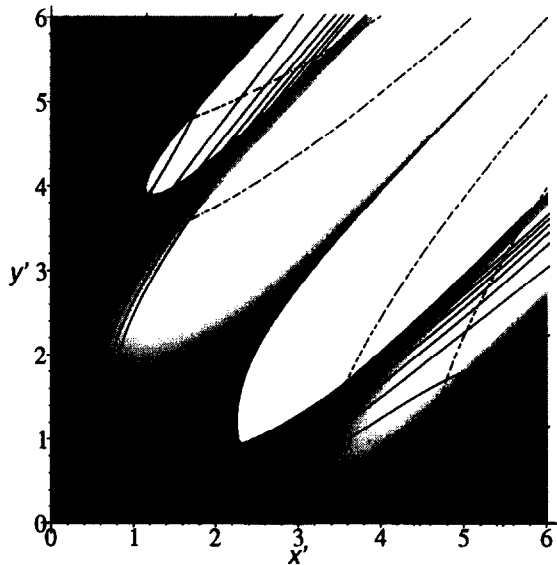
$$G(\xi, x) = \frac{\pi^2}{32} [(h_2(\xi) \text{sign}(\xi - x) - h_3(\xi))(g_1(x) - g_1(\xi)) + (h_1(\xi) \text{sign}(\xi - x) - h_3(\xi))(g_2(x) - g_2(\xi)) + (h_1(\xi) + h_2(\xi) - 2h_3(\xi) \text{sign}(\xi - x))(g_3(x) - g_3(\xi))], \tag{10}$$

where  $h_1 = |\xi| \mathcal{I}_+(\xi)^2$ ,  $h_2 = |\xi| \mathcal{I}_-(\xi)^2$  and  $h_3 = \xi \mathcal{I}_+(\xi) \mathcal{I}_-(\xi)$ , while  $g_i = \int_0^\xi h_i d\xi$ ,  $i = 1, 2, 3$ .

The constants  $c_i^j$  in the above solution may be chosen unambiguously for any different  $x_1$  and  $x_2$  so that the stream function will be continuous up to and including its third derivatives. The latter guarantees continuity of the pressure and thereby makes this family of exact solutions physically acceptable.



**Fig. 3.** The light and dark grey half-tones represent correspondingly the regions of positive and negative current density of a visco-resistive flow with stream lines shown by thick solid curves, while thin dashed curves represent the unperturbed magnetic field lines in the one quadrant of the  $(x', y')$ -plane; a sharp dark border of the current density distribution outlines the region of reverse current.



**Fig. 4.** The same as on Figure 3 stream and field line patterns superimposed on the pressure distribution whose variation is depicted by the corresponding distribution of grey half-tones: the light and dark ones correspond respectively to higher and lower values of pressure; a sharp border of this distribution outlines the regions of lowest pressure.

An example of such solutions with  $x_1 = 1.0$  and  $x_2 = 2.2933$  is shown in Figure 3 demonstrating that the current distribution has a rather nontrivial structure. In the central region there is a nearly axisymmetrical current spike which turns gradually into a region of reverse current when moving outward along the separatrices. This reverse current grows in value quadratically with increasing the distance from the centre. Physically, such structure of the separatrix layer allows the moving fluid to balance the viscous stress tangential to the separatrix with a corresponding pressure gradient (Figure 4), since the Lorentz force has no component along the separatrix. However, this produces also a normal pressure gradient which is counterbalanced by the resulting Lorentz force. Other examples and generalizations of this theory will be given in a following paper.

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