The structure of three-dimensional magnetic neutral points

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The local configurations of three-dimensional magnetic neutral points are investigated by a linear analysis about the null. It is found that the number of free parameters determining the arrangement of field lines is four. The configurations are first classified as either potential or non-potential. Then the non-potential cases are subdivided into three cases depending on whether the component of current parallel to the spine is less than, equal to or greater than a threshold current; therefore there are three types of linear non-potential null configurations (a radial null, a critical spiral and a spiral). The effect of the four free parameters on the system is examined and it is found that only one parameter categorizes the potential configurations, whilst two parameters are required if current is parallel to the spine. However, all four parameters are needed if there is current both parallel and perpendicular to the spine axis. The magnitude of the current parallel to the spine determines whether the null has spiral, critical spiral or radial field lines whilst the current perpendicular to the spine affects the inclination of the fan plane to the spine. A simple method is given to determine the basic structure of a null given **M** the matrix which describes the local linear structure about a null point. © *1996 American Institute of Physics.* [S1070-6634X(96)03803-4]

I. INTRODUCTION

Magnetic reconnection plays a central role in many phenomena that occur in plasmas. For example, in space, in the formation of X-ray bright points and solar flares on the Sun and in the interaction between the Earth's magnetosphere and the solar wind and, in the laboratory, in spheromaks. Over the last 20 years many aspects of two-dimensional reconnection have been extensively studied. In two dimensions the magnetic field vanishes at a neutral point which may be either the "X" type or "O" type. In three dimensions Refs. 1 and 2 have considered some aspects of magnetic reconnection at three-dimensional neutral points. We, in this paper, study such neutral points in detail by considering the local magnetic configurations that can occur around them.

To find the local magnetic structure about a neutral point we must consider the magnetic field in the neighbourhood of a point where the field vanishes ($\mathbf{B}=\mathbf{0}$). If, without loss of generality, we take the neutral point to be situated at the origin and, in addition, assume that the magnetic field approaches zero linearly, the magnetic field **B** near a neutral point may be expressed to lowest order as

$$\mathbf{B}=\mathbf{M}\cdot\mathbf{r}, \tag{1}$$

where **M** is a matrix with elements $M_{ij} = \partial B_i/\partial x_j$ and **r** is the position vector $(x,y,z)^T$. In this paper we systematically study the matrix **M**, first in two dimensions (Sec. II) as a preliminary to the three-dimensional work of Sec. III where the matrix **M** is reduced to its simplest three-dimensional form and the theory used in calculating the magnetic configurations is discussed. In Secs. IV and V we discuss the potential and non-potential configurations, respectively. Finally in Sec. VI this work is concluded.

II. REVIEW OF TWO-DIMENSIONAL NEUTRAL POINTS

In two dimensions the matrix **M** is simply

$$\mathbf{M}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where a_{ij} are real constants. The solenoidal constraint $\nabla\cdot\mathbf{B}=0$ gives $a_{11}=-a_{22}$: thus the trace of **M** is zero. The diagonal entries in the matrix are associated with the potential part of the field so we let $a_{11}=p$, and since the current associated with the neutral point is

$$\mathbf{J}=\frac{1}{\mu_0}(0,0,a_{21}-a_{12}),$$

we define

$$a_{12}=\frac{1}{2}(q-j_z) \quad \text{and} \quad a_{21}=\frac{1}{2}(q+j_z).$$

Clearly, for a current-free neutral point $a_{21}=a_{12}=q/2$, and the parameter q is therefore also associated with the potential field whilst j_z is the magnitude of the current perpendicular to the plane of the null point. The matrix **M** may now finally be written as

$$\mathbf{M}=\begin{bmatrix} p & \frac{1}{2}(q-j_z) \\ \frac{1}{2}(q+j_z) & -p \end{bmatrix}.$$

We will find it useful to define a threshold current,

$$j_{thresh}=\sqrt{4p^2+q^2}, \tag{2}$$

which only depends on the parameters associated with the potential part of the field, because $j_{thresh}=\sqrt{-4c}$, where

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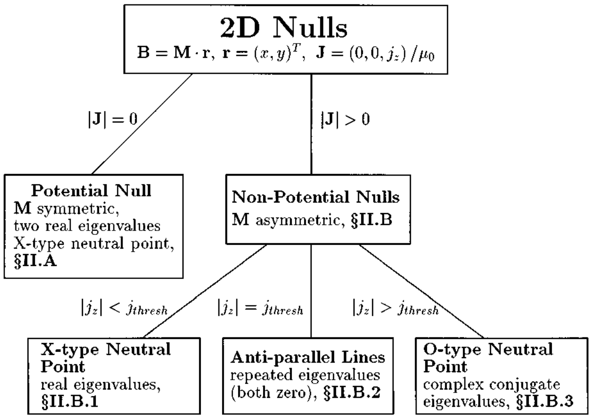


FIG. 1. A categorization of the different types of two-dimensional null and the respective limits of j_z (the z-component of current) and j_{thresh} (the threshold current) at which they occur.

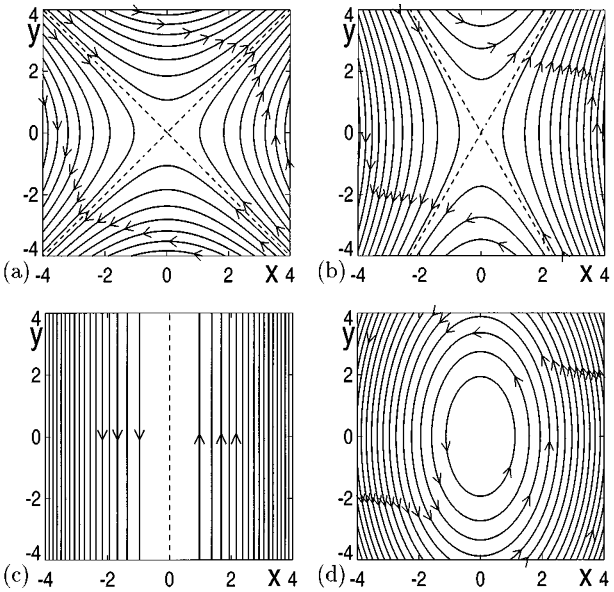


FIG. 2. Two-dimensional magnetic field plots of neutral points in the xy-plane showing (a) a potential X-point ($j_z=0$), (b) a non-potential X-point ($|j_z|<j_{thresh}$), (c) anti-parallel field lines ($|j_z|=j_{thresh}$) and (d) an elliptical null ($|j_z|>j_{thresh}$).

$\lambda^2+c=|\lambda\mathbf{I}-\mathbf{S}|=0$ and $\mathbf{S}$ is the symmetric part of $\mathbf{M}$. We now calculate the flux function A, which satisfies

$$B_X=\frac{\partial A}{\partial Y} \quad \text{and} \quad B_Y=-\frac{\partial A}{\partial X},$$

so that

$$A=\frac{1}{4}[(q-j_z)Y^2-(q+j_z)X^2]+pXY. \tag{3}$$

If we rotate the XY-axes through an angle θ to give xy-axes using the relations

$$\begin{aligned} X&=x\cos\theta-y\sin\theta,\\ Y&=x\sin\theta+y\cos\theta, \end{aligned} \tag{4}$$

and substitute (4) into (3) with

$$\tan 2\theta=-2\frac{p}{q},$$

with j_{thresh} as in (2), then A becomes

$$A=\frac{1}{4}((j_{thresh}-j_z)y^2-(j_{thresh}+j_z)x^2). \tag{5}$$

We thus see that in two dimensions the two parameters j_{thresh} and j_z govern the magnetic configuration.

The eigenvalues of the matrix $\mathbf{M}$ are given by

$$\lambda=\pm\frac{1}{2}\sqrt{j_{thresh}^2-j_z^2}; \tag{6}$$

hence, depending on whether the current j_z is greater or less than the threshold value j_{thresh} the eigenvalues will be real or imaginary and the field will have a different structure.

In the following sections a general two-dimensional null is studied first depending on whether it is potential (Sec. II A) or not (Sec. II B) and then whether the current is greater or less than j_{thresh} (Fig. 1).

A. Potential two-dimensional neutral points

In the case of a current-free two-dimensional null $j_z=0$, $\mathbf{M}$ is symmetric, and the eigenvalues are given by

$$\lambda=\pm\frac{j_{thresh}}{2};$$

therefore we have two real non-zero eigenvalues, and consequently Eq. (5) becomes

$$A=\frac{j_{thresh}}{4}(y^2-x^2).$$

The field lines are therefore rectangular hyperbola and the separatrices intersect at an angle of $\pi/2$. This is a potential *X-type neutral point*, as shown in Fig. 2(a), and is the only possible configuration for a current-free two-dimensional neutral point.

B. Non-potential two-dimensional neutral points

Two-dimensional neutral points with current are classified with respect to the magnitudes of j_z and j_{thresh}.

1. $|j_z|<j_{thresh}$

When $|j_z|<j_{thresh}$ the eigenvalues are real, equal in magnitude, but opposite in sign ($\det\mathbf{M}<0$). From the flux function we see that the field lines are hyperbolae with separatrices that intersect at an angle of

$$\tan^{-1}\left(\frac{(j_{thresh}^2-j_z^2)^{1/2}}{j_z}\right).$$

The null point formed is therefore an X-type neutral point as shown in Fig. 2(b). As $j_z\rightarrow 0$ the hyperbolae tend to rect-

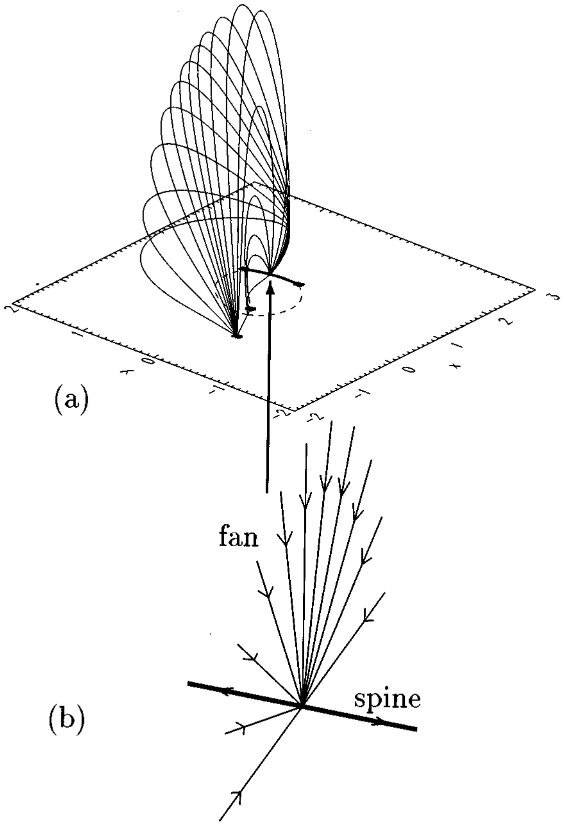


FIG. 3. (a) A three-dimensional potential configuration showing the global magnetic field structure due to four point sources (depicted by asterisks) containing two neutral points on the $z=0$-plane. (b) A schematic enlargement of one of the nulls showing the local structure about a three-dimensional negative neutral point with a fan (thin lines) and a spine (thick lines).

angular hyperbolae, thus reducing to the potential case. As the current increases, the angle between the separatrices increases as they close up along the y-axis.

2. $|j_z|=j_{thresh}$

The eigenvalues in this case are equal ($\det \mathbf{M}=0$), and so from Eq. (6) they must be equal to zero. The flux function depends on either just x^2 if $j_{thresh}=j_z$ or just y^2 if $j_{thresh}=-j_z$; thus the configuration contains anti-parallel field lines with a null line along the y-axis [Fig. 2(c)] or x-axis, respectively.

3. $|j_z|>j_{thresh}$

If $|j_z|>j_{thresh}$ the eigenvalues become complex conjugates ($\det \mathbf{M}>0$). When $j_{thresh}=0$, $p=q=0$ and the field configuration has circular field lines centered around the origin, whereas if $j_{thresh}\neq 0$ then the field contains concentric ellipses [Fig. 2(d)].

III. THEORY OF THREE-DIMENSIONAL NEUTRAL POINTS

Figure 3(a) shows a three-dimensional neutral point formed by a field due to four point sources, two positive and two negative.[3] If we look closely at the local structure near this null [Fig. 3(b)] we find that there is a set of field lines extending into the null point and forming a surface (the thin lines) which we call the *fan*, following the nomenclature of Priest and Titov.[2] However, only two field lines leave the null point (the thick lines) and they are called the *spine*. These are the two basic components that make up the skeleton of any neutral point in three dimensions. The *fan* is a surface made up of field lines which radiate out, or into the null point (this is the same as the Σ surface referred to by Cowley,[4] Greene[5] and Lau and Finn[1]). The *spine* is made up of two special field lines that are directed away from the null if the field lines in the fan are directed towards the null and vice-versa (these are equivalent to the γ line[4,5,1]). Field lines that lie near the null point, but do not pass through it, form bundles around the spine which spread out either side of the fan surface.

Mathematically, the linearised field about a three-dimensional neutral point may be described using Eq. (1) in terms of a 3×3 matrix of the form

$$\mathbf{M}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \tag{7}$$

where a_{ij} are real constants. The constraint $\nabla\cdot\mathbf{B}=0$ implies that the trace of $\mathbf{M}$ must be zero, giving

$$a_{11}+a_{22}+a_{33}=0\,.$$

This condition also implies that the eigenvalues λ_1,λ_2 and λ_3 associated with the matrix sum to zero. The eigenvectors associated with these eigenvalues are $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$.

If a magnetic field line near the null is written in terms of a position vector $\mathbf{r}=(x,y,z)^T$ which is dependent on an arbitrary parameter k then we may write

$$\frac{\mathrm{d}\mathbf{r}(k)}{\mathrm{d}k}=\mathbf{M}\cdot\mathbf{r}(k)=\mathbf{B}, \tag{8}$$

using the substitution $\mathbf{r}(k)=\mathbf{P}\mathbf{u}(k)$, where $\mathbf{P}$ is the matrix of the eigenvectors of $\mathbf{M}$, Eq. (8) becomes

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}k}=\mathbf{P}^{-1}\mathbf{M}\mathbf{P}\mathbf{u}. \tag{9}$$

There are now two cases we must consider depending on whether the matrix $\mathbf{M}$ can or can not be diagonalized. First, if $\mathbf{M}$ is diagonalizable to a matrix $\mathbf{\Lambda}$, say, which may have real or complex elements, then the above equation may be simply solved to give

$$\mathbf{u}=\mathbf{A}\,\exp(\mathbf{\Lambda}k),$$

where $\mathbf{A}$ is also a diagonal matrix with entries A,B and C which are constant along a field line, and implies

$$\mathbf{r}(k)=Ae^{\lambda_1 k}\mathbf{x}_1+Be^{\lambda_2 k}\mathbf{x}_2+Ce^{\lambda_3 k}\mathbf{x}_3\,. \tag{10}$$

Thus, each field line may be written in terms of the eigenvalues and eigenvectors of the matrix $\mathbf{M}$.

We initially consider the situation where all the eigenvalues are real. Since they sum to zero there is always one eigenvalue of opposite sign to the other two, say for example

TABLE I. Relation between the character of the eigenvalues and the associated vectors spanning the spine axis and fan plane.

Eigenvalues	Associated vectors
Three real and distinct	The associated eigenvectors $(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$.
Two repeated, one distinct	The three eigenvectors if they exist, or the vectors $(\mathbf{x}_1,\mathbf{x}_2^*,\mathbf{x}_3)$ which satisfy $\mathbf{M}\mathbf{x}_1=\lambda\mathbf{x}_1$, $\mathbf{M}\mathbf{x}_2^*=\lambda\mathbf{x}_2^*+\mathbf{x}_1$ and $\mathbf{M}\mathbf{x}_3=-2\lambda\mathbf{x}_3$.
Two complex conjugate, one real	If the eigenvectors are $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ then the vectors are $\mathbf{x}_1'=(\mathbf{x}_1+\mathbf{x}_2)/2$, $\mathbf{x}_2'=-\mathrm{i}(\mathbf{x}_1-\mathbf{x}_2)/2$, and $\mathbf{x}_3$.

$\lambda_1,\lambda_2>0,\lambda_3<0$. If we trace a field line backwards away from the neutral point, that is let $k\rightarrow-\infty$ in Eq. (10), we find

$$\mathbf{r}(k)\rightarrow Ce^{\lambda_3 k}\mathbf{x}_3,$$

so all the field lines that head in towards the null are parallel to the single eigenvector $\mathbf{x}_3$. However, if we trace forward along field lines away from the null, then $k\rightarrow\infty$ and

$$\mathbf{r}(k)\rightarrow Ae^{\lambda_1 k}\mathbf{x}_1+Be^{\lambda_2 k}\mathbf{x}_2.$$

This implies that the field lines that are directed away from the null lie parallel to the plane defined by the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. If we compare this with our geometrical understanding of a three-dimensional null then we find that the eigenvector $\mathbf{x}_3$ with negative eigenvalue λ_3 defines the path of the spine, whilst the plane of the fan is defined by the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

In the situation where we have two complex and one real eigenvalue, say $\eta\pm\mathrm{i}\nu$ and -2η, with corresponding eigenvectors $\mathbf{x}_1=(\mathbf{x}_1'+\mathrm{i}\mathbf{x}_2')/2$, $\mathbf{x}_2=(\mathbf{x}_1'-\mathrm{i}\mathbf{x}_2')/2$ and $\mathbf{x}_3$, respectively, then

$$\mathbf{r}(k)=\frac{1}{2}(A+\mathrm{i}B)e^{(\eta+i\nu)k}(\mathbf{x}_1'+\mathrm{i}\mathbf{x}_2')+\frac{1}{2}(A-\mathrm{i}B)\times e^{(\eta-i\nu)k}(\mathbf{x}_1'-\mathrm{i}\mathbf{x}_2')+Ce^{-2\eta k}\mathbf{x}_3,$$

where A, B and C are constant along a field line. This may be rewritten as

$$\mathbf{r}(k)=e^{\eta k}R\cos(\Theta k+\nu k)\mathbf{x}_1'-e^{\eta k}R\sin(\Theta k+\nu k)\mathbf{x}_2'+Ce^{-2\eta k}\mathbf{x}_3, \tag{11}$$

where A and B have been rewritten in terms of the constants R and Θ. So, if for example, we take $\eta>0$, then as $k\rightarrow\infty$ this equation reduces to

$$\mathbf{r}(k)\rightarrow Re^{\eta k}\cos(\Theta k+\nu k)\mathbf{x}_1'-Re^{\eta k}\sin(\Theta k+\nu k)\mathbf{x}_2'.$$

Thus the fan plane is defined by the vectors $\mathbf{x}_1'$ and $\mathbf{x}_2'$ and field lines in this plane will be spirals. The spine lies in the direction of the eigenvector $\mathbf{x}_3$, since, as $k\rightarrow-\infty$,

$$\mathbf{r}(k)\rightarrow Ce^{-2\eta k}\mathbf{x}_3.$$

The second case we must consider is when the matrix $\mathbf{M}$ is not diagonalizable. This occurs when two of the eigenvalues are repeated and the matrix can only reduce to a Jordan normal form $(\mathbf{J_n})$ which looks like

$$\mathbf{J_n}=\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 0\\ 0 & 0 & -2\lambda\end{bmatrix}.$$

The equation for the field lines may then be written using the substitution $\mathbf{r}(k)=\mathbf{P}\mathbf{u}(k)$, where this time $\mathbf{P}=(\mathbf{x}_1,\mathbf{x}_2^*,\mathbf{x}_3)$ and $\mathbf{x}_1$, $\mathbf{x}_2^*$ and $\mathbf{x}_3$ satisfy

$$\mathbf{M}\mathbf{x}_1=\lambda\mathbf{x}_1,\quad \mathbf{M}\mathbf{x}_2^*=\mathbf{x}_1+\lambda\mathbf{x}_2^*,\quad \mathbf{M}\mathbf{x}_3=-2\lambda\mathbf{x}_3, \tag{12}$$

such that

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}k}=\mathbf{J_n}\mathbf{u}.$$

The equation for a field line is therefore

$$\mathbf{r}(k)=(A+Bk)e^{\lambda k}\mathbf{x}_1+Be^{\lambda k}\mathbf{x}_2^*+Ce^{-2\lambda k}\mathbf{x}_3, \tag{13}$$

where A,B and C are all constant along a field line. Hence, if we assume $\lambda>0$ then running forwards along a field line so that $k\rightarrow\infty$ we find

$$\mathbf{r}(k)\rightarrow(A+Bk)e^{\lambda k}\mathbf{x}_1+Be^{\lambda k}\mathbf{x}_2^*;$$

thus the field lines lie in planes parallel to $\mathbf{x}_1$ and $\mathbf{x}_2^*$ whereas if we trace backwards along a field line then $k\rightarrow-\infty$ and

$$\mathbf{r}(k)\rightarrow Ce^{-2\lambda k}\mathbf{x}_3,$$

so that the field lines become parallel to the vector $\mathbf{x}_3$. We therefore find that the spine is defined by the eigenvector relating to the single eigenvalue whereas the fan plane is defined by the remaining eigenvector and the Jordan basis vector which are related to the repeated eigenvalue.

In general, therefore, the spine lies along the eigenvector of $\mathbf{M}$ that relates to the single eigenvalue whose sign is opposite to that of the real parts of the remaining eigenvalues. These remaining eigenvalues have vectors associated with them which define the fan plane and which depend on the nature of the eigenvalues, as shown in Table I.

It may easily be proved that if the real parts of two of the three eigenvalues are positive, say $\lambda_1,\lambda_2>0$, then the neutral point will have field lines in the fan directed away from the null and a spine pointing into the null along $\mathbf{x}_3$. This type of null is called a *positive neutral point*.[2] The determinant of the matrix $\mathbf{M}$ will always be negative for this type of null. However, if the real parts of two of the eigenvalues are negative, say $\lambda_1,\lambda_2<0$, then the fan plane will have field lines pointing into the null with the spine again lying along the eigenvector $\mathbf{x}_3$, but this time directed away from the null. Not surprisingly, this type of neutral point is called a *negative neutral point* and has determinant $\mathbf{M}$ greater than zero.

A. Reduction of M to its simplest form

In order to examine all possible configurations of the localised field about the neutral point we reduce $\mathbf{M}$ to the least number of free parameters. In doing so, it is important to remember that the matrix **M** determines all the physical characteristics of the field including its structure, current and associated Lorentz force. Thus, we do not consider the simplest mathematical form of **M** but derive a form for the matrix that gives the simplest topological form for the null without loss of generality. First we note that the field always has at least one real eigenvalue whose sign will always be opposite to the real parts of the other two. We therefore choose the local orthogonal coordinate system such that the eigenvector corresponding to this eigenvalue is in the z-direction, so that the spine is directed along the z-axis. Additionally, the matrix may be further reduced by rotating the xy-plane so that the new x-axis lies in the direction of the resultant current in the xy-plane. Finally, by dividing by a scaling factor the matrix reduces to

$$\mathbf{M}=\begin{bmatrix} 1 & \frac{1}{2}(q-j_\parallel) & 0 \\ \frac{1}{2}(q+j_\parallel) & p & 0 \\ 0 & j_\perp & -(p+1) \end{bmatrix}, \tag{14}$$

where $p \geqslant -1$ and $q^2 \leqslant j_\parallel^2+4p$. The potential part of the configuration is defined by the parameters p and q with the current given by

$$\mathbf{J}=\frac{1}{\mu_0}(j_\perp,0,j_\parallel), \tag{15}$$

where $j_\parallel$ is the component of current parallel to the spine and $j_\perp$ is the component of current perpendicular to the spine. Another way of producing the form (14) is by first splitting the matrix (7) into symmetric (**S**) and asymmetric (**A**) parts. Next diagonalize **S** so that the axes are along the eigenvectors then rotate about the z-axis to make the x-axis along $j_\perp$. Finally, rotate about the y-axis so that the upper half $y-z$ terms in **S** and **A** cancel. Note that the usual reduction of an arbitrary matrix to block diagonal form [i.e., (14) with $q=j_\perp=0$] does not allow all the possible field configurations.

Finally, we shall define a threshold current j_{thresh} which depends purely on p and q the potential field parameters such that

$$j_{thresh}=\sqrt{(p-1)^2+q^2}. \tag{16}$$

Again, similar to the two-dimensional case, $j_{thresh}=\sqrt{b^3/27+c^2/4}$, where $\lambda^3+b\lambda+c=|\lambda\mathbf{I}-\mathbf{S}|=0$. This implies that the three eigenvalues $(\lambda_1,\lambda_2,\lambda_3)$ associated with **M** may be written as

$$\lambda_1=\frac{p+1+\sqrt{j_{thresh}^2-j_\parallel^2}}{2},$$

$$\lambda_2=\frac{p+1-\sqrt{j_{thresh}^2-j_\parallel^2}}{2}, \quad \lambda_3=-(p+1). \tag{17}$$

We see, similar to the two-dimensional case, that it is the relative sizes of j_{thresh} and $j_\parallel$ which determine the nature of the eigenvalues and consequently the local magnetic configuration about the null point.

Note that in situations where $j_\perp$ equals zero the perpendicular component of current does not exist; therefore we have one further degree of freedom and can rotate the matrix about the spine (z-axis) such that it reduces to the form

$$\mathbf{M}=\begin{bmatrix} 1 & -\frac{1}{2}j_\parallel & 0 \\ \frac{1}{2}j_\parallel & p & 0 \\ 0 & 0 & -(p+1) \end{bmatrix};$$

thus, throughout this paper in studying configurations where $j_\perp=0$ we assume $q=0$ without loss of generality. Also note that having taken a scaling factor from our matrix we are excluding the possibility of all the elements of the trace equaling zero. This is of course a special situation and in general does not arise; however, for completeness we do mention such situations when they arise. Further, it is worth mentioning that since $p \geqslant -1$ the eigenvalue relating to the spine is always negative; hence all the three-dimensional configurations we consider in Secs. IV and V of this paper are positive neutral points. This means that all the field lines in the fan planes are emanating outwards and the spines are composed of pairs of field lines directed towards the origin.

We now briefly mention some of the previous work on three-dimensional neutral points. Cowley[4] studied a current-free neutral point of the form

$$\mathbf{B}=(\alpha x,\beta y,-(\alpha+\beta)z),$$

where α and β are of the same sign, with eigenvalues α,β and $-(\alpha+\beta)$. Cowley referred to the case $\alpha,\beta>0$ as Type A, which we call a *positive radial null*. Similarly, the case $\alpha,\beta<0$ was referred to as Type B and is a *negative radial null* point. For $\alpha=\beta$ field lines from the neutral point radiate with equal spacing from the null; thus we call this a *proper radial null*, whereas for $\alpha \neq \beta$ the field lines of the null are unevenly spaced and are orientated in one preferential direction, known here as the *major axis of the fan*; thus the null is called an *improper radial null*.

Fukao *et al.*[6] studied more general neutral points than Cowley.[4] They considered a 3×3 matrix containing six parameters and found that when all three eigenvalues are real the null point is radial, but one real and two complex conjugate give field lines that spiral logarithmically in the fan plane—this type of null is known as a *proper spiral null*. They found that if there is no fan current the spine and fan are perpendicular. Two-dimensional neutral points were also found in special circumstances containing either a line of X-points or O-points, depending on whether the eigenvalues are real or imaginary.

In this paper we extend the previous work undertaken on three-dimensional magnetic neutral point structure by studying comprehensively the most general form for the matrix **M** which defines the local magnetic field about the null. With our matrix all linear magnetic field configurations that can

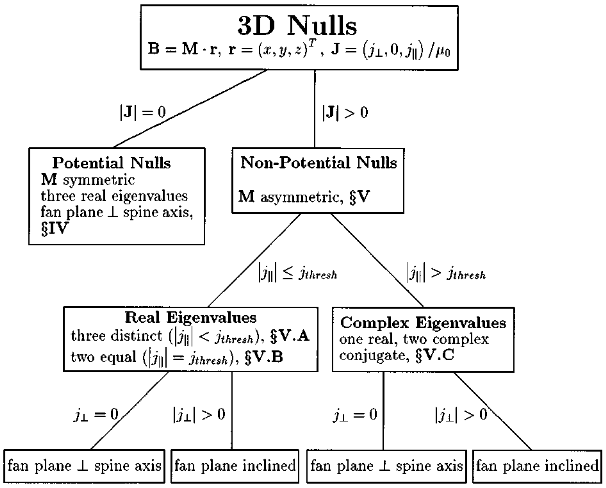


FIG. 4. A categorization of the various types of three-dimensional neutral points with respect to the relative sizes of $j_\parallel$ (the component of current parallel to the spine of the null) and j_{thresh} (the threshold current).

arise are studied and at most two rotations and a multiplication by a scalar are needed simply to transform our set of linear null points into any other. Through our study, which follows a similar procedure to that undertaken in studying the two-dimensional neutral point in Sec. II, we note that there are in fact extra configurations to those previously discovered. For instance, we find that logarithmically spiraling nulls are only one special case of the family of spiraling nulls, and, in general, field lines in the fan of a spiral form a more complex spiraling pattern; these nulls are known as *improper spiral nulls*. Also, we can explain how the current of a neutral point determines its structure. Figure 4 illustrates how the three-dimensional null point structures may be divided with respect to the magnitudes of the components of current and indicates what types of null are studied in the following sections.

In the next two sections all the three-dimensional figures are illustrated as follows. The spine is plotted as a solid thick line in the z-direction. The fan plane is shown by the square region enclosed by dashed lines with the fan field lines themselves depicted by continuous lines. A bundle of field lines around the spine is illustrated by dashed lines and are drawn only below the fan plane for clarity.

IV. THREE-DIMENSIONAL POTENTIAL NULLS

The matrix **M** representing the linear field about a potential three-dimensional null is symmetric and may be written as

$$\mathbf{M}=\begin{bmatrix}1 & 0 & 0\\ 0 & p & 0\\ 0 & 0 & -(p+1)\end{bmatrix}.$$

The eigenvalues relating to this matrix are

$$\lambda_1=1,\quad \lambda_2=p,\quad \lambda_3=-(p+1).$$

By our choice of the matrix **M** we find we must have $p\geqslant 0$ such that the eigenvalue $-(p+1)$ which corresponds to the eigenvector that lies along the z-axis forms the spine of the neutral point, as required. The eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are

$$\mathbf{x}_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad \mathbf{x}_2=\begin{pmatrix}0\\1\\0\end{pmatrix},\quad \mathbf{x}_3=\begin{pmatrix}0\\0\\1\end{pmatrix},$$

thus, as found by Fukao *et al.*,[6] the fan plane is perpendicular to the spine in a potential situation.

The threshold current $j_{thresh}=|p-1|$ in this situation. Depending on its value we have three cases to consider. First, when $j_{thresh}=0$ and $p>0$ all the eigenvalues are nonzero, but two are equal (Sec. IV A). Second, in Sec. IV B we examine the situation where all the eigenvalues are non-zero and unequal ($p>0$, $j_{thresh}>0$), and finally we have the case where one eigenvalue is zero, $p=0$ (Sec. IV C).

A. $p>0$, $j_{thresh}=0$

Assuming $p>0$ and $j_{thresh}=0$, the only value that the parameter p can take is $p=1$ so that

$$\mathbf{M}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}.$$

We find now that two of the eigenvalues are repeated; however, three eigenvectors may still be found and are the same as in the previous case. The field associated with this matrix is a positive proper radial null, as depicted in Fig. 5(a). Here, since this is the first three-dimensional null illustrated in this paper, we also draw the field lines in the fan plane (xy-plane) [Fig. 5(b)] and the field lines in the xz-plane which is perpendicular to the surface of the fan [Fig. 5(c)].

B. $p>0$, $j_{thresh}>0$

If $p>0$ and $j_{thresh}>0$ then improper radial nulls are formed with the field aligned predominantly in the direction of the eigenvector corresponding to the greatest eigenvalue of the two associated with the fan plane. The field lines rapidly curve such that they run parallel to the x-axis if $0<p<1$ and parallel to the y-axis if $p>1$ [see Figs. 5(d) and 5(e), respectively]. This is as predicted, for if we consider Eq. (10) for a field line in this particular case, we find that as $k\rightarrow\infty$,

$$r(k)\rightarrow Ae^{k}\mathbf{x}_1+Be^{pk}\mathbf{x}_2.$$

So if $0<p<1$ then $r(k_\infty)\approx Ae^{k}\mathbf{x}_1$ and the field lines lie in the xy-plane, but are inclined along the major fan axis $y=0$, whereas if $p>1$ then $r(k_\infty)\approx Be^{pk}\mathbf{x}_2$ and the major fan axis of the improper null is the $x=0$ line.

C. $p=0$

When $p=0$, the matrix **M** reduces to

$$\mathbf{M}=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix},$$

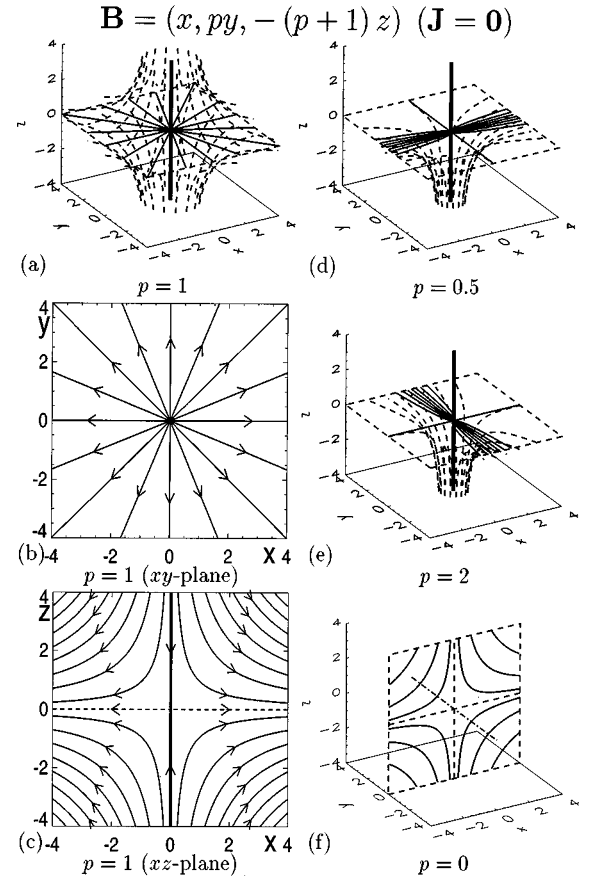


FIG. 5. The magnetic field configurations of three-dimensional potential fields. (a) The complete three-dimensional structure of a radial null ($p=1$), showing the field lines in (b) the fan plane (xy-plane) and (c) the xz-plane. An improper radial null with field lines aligned along (d) the x-axis and (e) the y-axis. (f) When $p=0$ the null point reduces to a two-dimensional potential X-point field with the y-axis a null line (dot-dashed).

thus the field lines become two-dimensional potential X-points lying in planes parallel to the xz-plane and form a null-line along the y-axis [Fig. 5(f)]. Note that if a scaling factor had not been taken from the matrix the only extra possible field line configuration is the trivial situation of $\mathbf{B}=\mathbf{0}$.

V. THREE-DIMENSIONAL NON-POTENTIAL NULLS

Here the matrix $\mathbf{M}$ is asymmetric and has an associated current $\mathbf{J}=(j_\perp,0,j_\parallel)$. The eigenvalues of the matrix $\mathbf{M}$ are

$$\lambda_{1,2}=\frac{1}{2}(p+1)\pm\frac{1}{2}\sqrt{j_{thresh}^2-j_\parallel^2}, \quad \lambda_3=-(p+1), \quad (18)$$

where $p\geqslant -1$, $(p+1)^2\geqslant j_{thresh}^2-j_\parallel^2$ and $j_{thresh}^2=(p-1)^2+q^2$, as previously defined. These constraints are necessary to ensure that the eigenvalue λ_3 always corresponds to the eigenvector that defines the spine of the null.

A. $|j_\parallel|<j_{thresh}$

First, let us consider the situation where the magnitude of the component of current parallel to the spine is less than that of the threshold current. This implies that all three eigenvalues are real and distinct and all three eigenvectors exist.

1. $j_\perp=0$ and $j_\parallel\neq 0$

The perpendicular component of current is zero here so we may assume $q=0$ and if $p>-j_\parallel^2/4$ the eigenvectors are found to be

$$\mathbf{x}_{1,2}=\begin{pmatrix} \dfrac{1-p\pm\sqrt{j_{thresh}^2-j_\parallel^2}}{j_\parallel} \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

so the fan and spine are perpendicular [Fig. 6(a)]. Substituting λ_i and $\mathbf{x}_i$ into Eq. (10) we find that as $k\rightarrow\infty$ the field lines in the plane of the fan become parallel to the line,

$$y=\frac{1-p-\sqrt{j_{thresh}^2-j_\parallel^2}}{j_\parallel}\,x.$$

The field lines in the fan do not form the same sort of improper null as found in the potential situation but instead form a skewed improper null since the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are not perpendicular. When $p=-j_\parallel^2/4$ the field reduces to a two-dimensional configuration [Fig. 6(b)] containing X-points in planes parallel to the plane,

$$j_\parallel x-2y=0,$$

and a null line (dot-dashed) along

$$y=\frac{2x}{j_\parallel}.$$

2. $j_\perp\neq 0$ and $j_\parallel=0$

If the current is purely perpendicular to the spine then $j_\perp\neq 0$ and $j_\parallel=0$, and the eigenvalues for $\mathbf{M}$ simply become

$$\lambda_{1,2}=\frac{1}{2}(p+1\pm j_{thresh}), \quad \lambda_3=-(p+1),$$

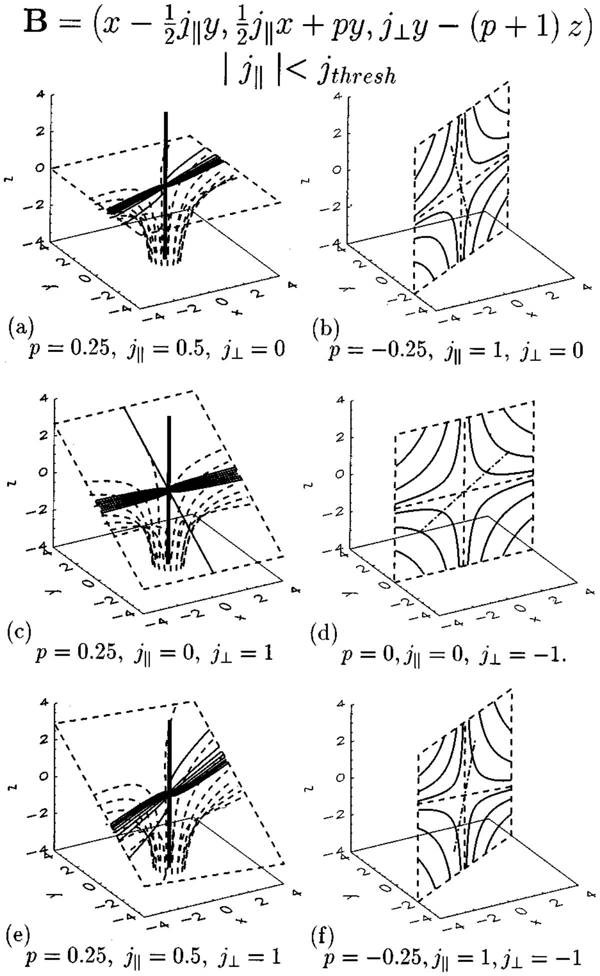
where $p\geqslant j_{thresh}-1$. If $p>j_{thresh}-1$ the eigenvectors are given by

$$\mathbf{x}_{1,2}=\begin{pmatrix} \dfrac{-3p^2+3+j_{thresh}^2\pm 2(p+2)j_{thresh}}{2j_\perp\sqrt{j_{thresh}^2-(p-1)^2}} \\ \dfrac{3+3p\pm j_{thresh}}{2j_\perp} \\ 1 \end{pmatrix},$$

$$\mathbf{x}_3=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The plane of the fan is therefore not perpendicular to the spine [Fig. 6(c)] and is defined by the equation

$$2j_\perp\sqrt{j_{thresh}^2-(p-1)^2}\,x-4j_\perp(p+2)y+\big(9(p+1)^2-j_{thresh}^2\big)z=0\,.$$



(a) $p=0.25,\ j_\parallel=0.5,\ j_\perp=0$ (b) $p=-0.25,\ j_\parallel=1,\ j_\perp=0$

(c) $p=0.25,\ j_\parallel=0,\ j_\perp=1$ (d) $p=0, j_\parallel=0,\ j_\perp=-1.$

(e) $p=0.25,\ j_\parallel=0.5,\ j_\perp=1$ (f) $p=-0.25, j_\parallel=1, j_\perp=-1$

FIG. 6. Non-potential three-dimensional magnetic field configurations where the magnitude of the current parallel to the spine is less than the threshold current. (a) and (b) $j_\perp=0, j_\parallel\neq 0$ and either $p>-j_\parallel^2/4$ or $p=-j_\parallel^2/4$, respectively. (c) and (d) $j_\perp\neq 0, j_\parallel=0$ and either $p>j_{thresh}-1$ or $p=j_{thresh}-1$, respectively. (e) and (f) $j_\perp\neq 0, j_\parallel\neq 0$ and either $p>-1$ or $p=\sqrt{j_{thresh}^2-j_\parallel^2}-1$, respectively. (b), (d) and (f) all have a line of null points indicated by a dot-dashed line.

Note that, as $j_\perp$ increases, the angle between the fan and spine reduces so that ultimately ($j_\perp\to\infty$) the spine lies in the fan plane. Also note that the fan does not necessarily tilt about the x-axis (the direction of the current) and so the perpendicular component of current does not in general lie in the plane of the fan. The field lines in the fan are positive improper nulls which orientate themselves predominantly along the line

$$\mathbf{l}(\gamma)=\left(\frac{-3p^2+3+j_{thresh}^2+2(p+2)j_{thresh}}{2j_\perp\sqrt{j_{thresh}^2-(p-1)^2}}\gamma,\right.$$
$$\left.\times\frac{3+3p+j_{thresh}}{2j_\perp}\gamma,\gamma\right),$$

where γ is real. This line is associated with the eigenvalue λ_1 since $\lambda_1>\lambda_2$. When $p=j_{thresh}-1$ the field reduces to a two-dimensional situation [Fig. 6(d)] where successive X-points form in xz-planes with a null line along

$$\mathbf{l}(\gamma)=\left(\frac{-(p+1)\sqrt{p}}{j_\perp}\gamma, \frac{p+1}{j_\perp}\gamma,\gamma\right).$$

3. $j_\perp\neq 0$ and $j_\parallel\neq 0$

When $p>-1$ and $(p+1)^2>j_{thresh}^2-j_\parallel^2$ and there are both parallel and perpendicular components of current, the eigenvalues are as in Eq. (18) and have corresponding eigenvectors,

$$\mathbf{x}_{1,2}=\begin{pmatrix}\dfrac{-3p^2+3+j_{thresh}^2-j_\parallel^2\pm 2(p+2)\sqrt{j_{thresh}^2-j_\parallel^2}}{2j_\perp(\sqrt{j_{thresh}^2-(p-1)^2}+j_\parallel)}\\ \dfrac{3(p+1)\pm\sqrt{j_{thresh}^2-j_\parallel^2}}{2j_\perp}\\ 1\end{pmatrix},$$

$$\mathbf{x}_3=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

The fan is therefore tilted towards the spine and lies in the plane

$$2j_\perp(\sqrt{j_{thresh}^2-(p-1)^2}+j_\parallel)x-4j_\perp(p+2)y+(9(p+1)^2$$
$$-j_{thresh}^2+j_\parallel^2)z=0;$$

thus the current perpendicular to the spine does not lie in the plane of the fan. The field lines lying in the fan plane form a positive skewed improper null [Fig. 6(e)] whose major fan axis is in the direction of the vector $\mathbf{x}_1$ since $\lambda_1>\lambda_2$, and thus the term $Ae^{\lambda_1 k}\mathbf{x}_1$ dominates in the equation for a field line in the fan. The field reduces to a two-dimensional X-point when $(p+1)^2=j_{thresh}^2-j_\parallel^2$ [Fig. 6(f)]. The plane of the X-point is

$$2px-(\sqrt{4p+j_\parallel^2}-j_\parallel)y=0,$$

and the null line lies along

$$\mathbf{l}(\gamma)=\left(\frac{-(p+1)(\sqrt{4p-j_\parallel^2}-j_\parallel)}{2j_\perp}\gamma, \frac{p+1}{j_\perp}\gamma,\gamma\right).$$

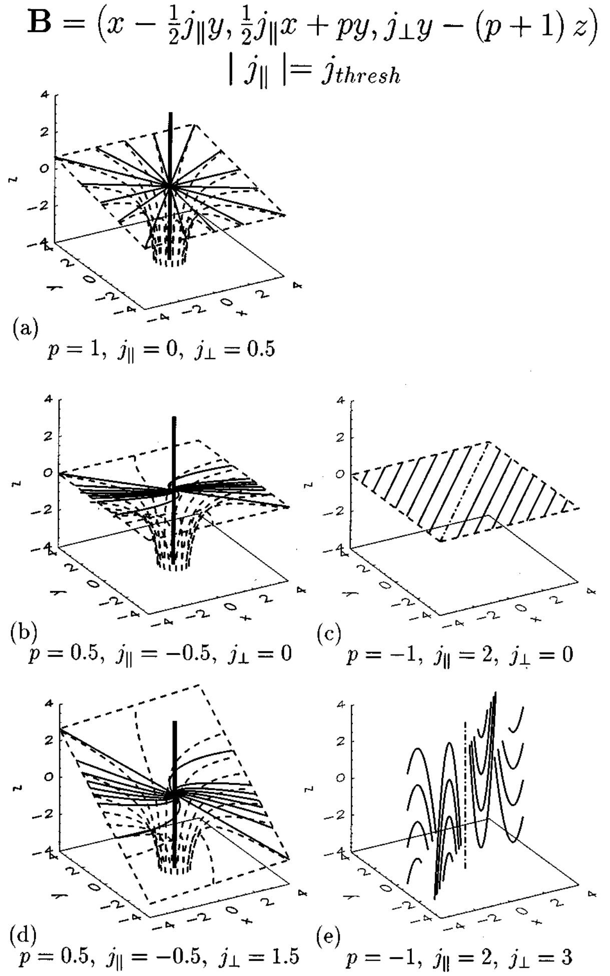
B. $|j_\parallel|=j_{thresh}$

In the case where $|j_\parallel|=j_{thresh}$ we find that two of the eigenvalues are repeated so that, with $p\geqslant -1$,

$$\lambda_{1,2}=\frac{p+1}{2}\quad\text{and}\quad\lambda_3=-(p+1).$$

1. $j_\perp\neq 0$ and $j_\parallel=0$

If the component of the current parallel to the spine is zero ($j_\parallel=j_{thresh}=0$), then we must have $p=1$ and $q=0$ since we require the z-axis to be the spine. The eigenvalues are therefore $1, 1, -2$ and the corresponding eigenvectors are, respectively,



(a) $p = 1,\ j_{\|} = 0,\ j_{\perp} = 0.5$

(b) $p = 0.5,\ j_{\|} = -0.5,\ j_{\perp} = 0$

(c) $p = -1,\ j_{\|} = 2,\ j_{\perp} = 0$

(d) $p = 0.5,\ j_{\|} = -0.5,\ j_{\perp} = 1.5$

(e) $p = -1,\ j_{\|} = 2,\ j_{\perp} = 3$

FIG. 7. Non-potential three-dimensional magnetic field structures for situations where the magnitude of the current parallel to the spine is equal to the threshold value $|p-1|$. (a) $j_{\perp} \neq 0$, $j_{\|}=0$ and $p=1$. (b) and (c) $j_{\perp}=0$, $j_{\|} \neq 0$ and either $p>-1$ or $p=-1$, respectively. (d) and (e) $j_{\perp} \neq 0$, $j_{\|} \neq 0$ and either $p>-1$ or $p=-1$, respectively. In (c) the plane $x=y{:}z$ is a null plane and in (e) the z-axis is a null line (shown by dot-dashed lines).

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ \dfrac{3}{j_{\perp}} \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since the null is non-potential the component of current perpendicular to the spine is non-zero. Thus, the fan does not lie in the xy-plane, but is in fact defined by the equation

$$j_{\perp}y - 3z = 0\,.$$

The field lines lying in the plane of the fan extend radially outwards and form what looks like a radial null [Fig. 7(a)].

2. $j_{\perp} = 0$ and $j_{\|} \neq 0$

When $p \geqslant -1$ and $|j_{\|}| = j_{thresh} \neq 0$ we find that the two repeated eigenvalues have only one associated eigenvector, so to define the plane of the fan an extra vector must be calculated, known as a Jordan basis vector. This is found by solving

$$\mathbf{M}\mathbf{x}_2^* = \lambda \mathbf{x}_2^* + \mathbf{x}_1\,,$$

where λ is the repeated eigenvalue and $\mathbf{x}_1$ is its associated eigenvector (see Sec. III).

If $p>-1$ ($p \neq 1$) and the perpendicular current is zero we may assume $q=0$ so the vectors which define the fan plane and spine of the null point are found to be

$$\mathbf{x}_1 = \begin{pmatrix} \dfrac{1-p}{j_{\|}} \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2^* = \begin{pmatrix} \dfrac{3-p}{j_{\|}} \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};$$

thus the fan plane is perpendicular to the spine. We find that the neutral point has a new form. It is neither an improper null nor, because of the straight lines in the fan, is it a spiral, and so we call it a *critical spiral* [Fig. 7(b)].

The eigenvector $\mathbf{x}_1$ is dominant in Eq. (13) as $k \to \infty$ for a field line in the fan; therefore field lines will orientate themselves towards the line of this vector as they move farther from the null.

If $p=-1$ then the neutral point merely reduces to a two-dimensional non-potential null with anti-parallel field lines such that the $x=y{:}z$-plane becomes a null plane [Fig. 7(c)].

3. $j_{\perp} \neq 0$ and $j_{\|} \neq 0$

When $p>-1$, $j_{\perp} \neq 0$ and the parallel component of the current is non-zero, but still equal to the threshold value, we again have repeated eigenvalues and have to look for a Jordan basis vector. The vectors are found to be

$$\mathbf{x}_1 = \begin{pmatrix} \dfrac{3(p+1)\left(\sqrt{j_{\|}^2-(p-1)^2}-j_{\|}\right)}{2j_{\perp}(p-1)} \\ \dfrac{3(p+1)}{2j_{\perp}} \\ 1 \end{pmatrix},$$

$$\mathbf{x}_2^* = \begin{pmatrix} \dfrac{(3p^2-4p-11)\left(\sqrt{j_{\|}^2-(p-1)^2}-j_{\|}\right)}{2j_{\perp}(p-1)^2} \\ \dfrac{3p+5}{2j_{\perp}} \\ 1 \end{pmatrix},$$

$$\mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

As in the previous cases we find that a critical spiral [Fig. 7(d)] is created whose major fan axis is in the direction of the eigenvector $\mathbf{x}_1$, but this time the fan lies in the plane,

$$2j_{\perp}\left(\sqrt{j_{\|}^2-(p-1)^2}+j_{\|}\right)x - 4j_{\perp}(p+2)y + 9(p+1)^2 z = 0\,.$$

If $p=-1$ then all the eigenvalues are zero parabolic field lines formed lying in parallel $x=y+\text{constant}{:}z$-planes [Fig. 7(e)]; close field lines all have their turning points along the $y=0$ line. The z-axis becomes a null line with anti-parallel field lines lying in the $x=y$:z-plane.

C. $|j_{\|}|>j_{thresh}$

When the parallel component of current is greater in magnitude than the threshold current two of the eigenvalues of **M** will be complex conjugates,

$$\lambda_{1,2}=\frac{p+1}{2}\pm\frac{\mathrm{i}}{2}\sqrt{j_{\|}^2-j_{thresh}^2},\quad \lambda_3=-(p+1).$$

Obviously the eigenvectors relating to the complex eigenvalues will also be complex conjugates; however, we have already seen in Sec. III that if the complex vectors are split up into their real and imaginary parts then these resultant vectors define the plane of the fan.

1. $j_{\perp}=0$ and $j_{\|}\neq 0$

Zero perpendicular current gives rise to a spine perpendicular to the fan consistent with the previous cases and has basis vectors,

$$\mathbf{x}_1'=\begin{pmatrix}\dfrac{1-p}{j_{\|}}\\ 1\\ 0\end{pmatrix},\quad \mathbf{x}_2'=\begin{pmatrix}\dfrac{\sqrt{j_{\|}^2-j_{thresh}^2}}{j_{\|}}\\ 0\\ 0\end{pmatrix},\quad \mathbf{x}_3=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix},$$

where it is assumed that $q=0$ and $p>-1$. The field lines in the fan plane form spirals of the form

$$\rho=\frac{C}{\sqrt{(p-1)\sin 2\phi+j_{\|}}}\times\exp\left(\frac{(p+1)\tan^{-1}[(j_{\|}\tan\phi+p-1)/\sqrt{j_{\|}^2-(p-1)^2}]}{\sqrt{j_{\|}^2-(p-1)^2}}\right),\tag{19}$$

where $\rho=\sqrt{x^2+y^2}$, $\phi=\tan^{-1}y/x$ and C is an arbitrary constant. Remembering that $j_{thresh}=|p-1|$ in this case we find that no singularities arise as along as the condition $|j_{\|}|>j_{thresh}$ for a spiral holds. These are in general not logarithmic spirals [Figs. 8(a) and 8(c)] contrary to Refs. 6 and 7. Logarithmic spirals only occur when $p=1$ [Fig. 8(b)], such that Eq. (19) reduces to

$$\rho=D\exp\left(\frac{2\phi}{j_{\|}}\right),$$

where D is an arbitrary constant. Note also that the associated vectors for the null are perpendicular if and only if $p=1$. Some of the spirals are so weakly oscillating that they look more like improper nulls [Fig. 8(a)], whereas others are tightly coiled [Fig. 8(b)]. If we look at Eq. (11) for any general field line (not necessarily in the plane of the fan) we can easily see that field lines oscillates in the $\mathbf{x}_1'$ and $\mathbf{x}_2'$ directions, and so they spiral around the spine until they spread spiraling outwards parallel to the fan plane [Fig. 8(d)].

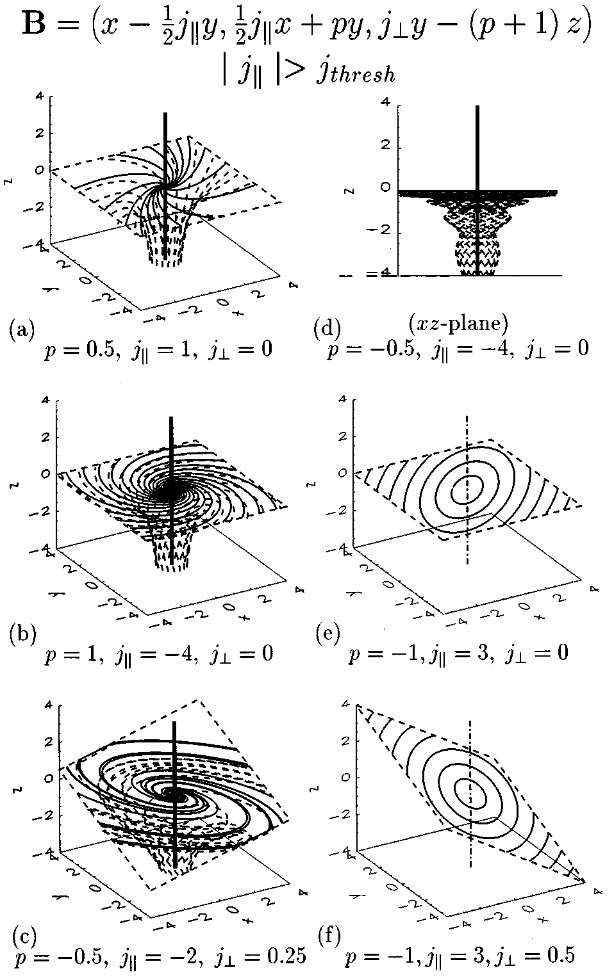


FIG. 8. Magnetic field configurations of three-dimensional non-potential neutral points, where the magnitude of the current parallel to the spine is greater than the threshold value. (a) and (b) $j_{\perp}=0,j_{\|}\neq 0$. (c) $j_{\perp}\neq 0,j_{\|}\neq 0$. (d) $j_{\perp}=0,j_{\|}\neq 0$ (xz-plane). (e) $j_{\perp}=0,j_{\|}\neq 0$ and $p=-1$. (f) $j_{\perp}\neq 0,j_{\|}\neq 0$ and $p=-1$. In (e) and (f) the null line is indicated by a dot-dashed line.

When $p=-1$ the field reduces to a two-dimensional null with a null line along the z-axis and elliptical field lines in successive z=constant-planes [Fig. 8(e)]. These elliptical field lines would become circular if the matrix had zero entries along the trace, i.e., if we had not taken a scaling factor from the matrix.

2. $j_{\perp}\neq 0$ and $j_{\|}\neq 0$

It is not possible to create a spiral null without a component of current parallel to the spine, so we next consider $j_{\perp}\neq 0$ and $j_{\|}\neq 0$. Basis vectors for this situation are found to be

$$\mathbf{x}_1'=\begin{pmatrix}\dfrac{-3p^2+3+j_{thresh}^2-j_{\|}^2}{2j_{\perp}(\sqrt{j_{thresh}^2-(p-1)^2}+j_{\|})}\\ \dfrac{3(p+1)}{2j_{\perp}}\\ 1\end{pmatrix},$$

TABLE II. Rules determining the three-dimensional structure of a magnetic null with field $\mathbf{B}=\mathbf{M}\cdot\mathbf{r}$. The characteristic equation of $\mathbf{M}$ is of third order and can always be written as $\lambda^3+Q\lambda+R=0$. The discriminant is then defined as $Q^3/27+R^2/4$ and the determinant as $-R$.

Discriminant	$\mathbf{M}$ symmetric ?	Determinant	Type of null
<0 $(j^2_{thresh}-j^2_\parallel>0)$	yes⇒potential	$\neq 0$, ⇒3-D null (<0 positive, >0 negative)	Improper radial null (fan $\perp$ spine)
		$=0$, ⇒2-D null	Continuous potential X-points ($\perp$ to null line)
	no⇒non-potential	$\neq 0$, ⇒3-D null (<0 positive, >0 negative)	Skewed improper null (fan $\perp$ spine if $j_\perp=0$) (fan $\not\perp$ spine if $j_\perp\neq 0$)
		$=0$, ⇒2-D null	Continuous X-points ($\perp$ to null line if $j_\perp=0$) ($\not\perp$ to null line if $j_\perp\neq 0$)
$=0$ $(j^2_{thresh}-j^2_\parallel=0)$	yes⇒potential	$\neq 0$, ⇒3-D null (<0 positive, >0 negative)	Proper radial null (fan $\perp$ spine)
	no⇒non-potential	$\neq 0$, ⇒3-D null (<0 positive, >0 negative)	Critical spiral null (fan $\perp$ spine if $j_\perp=0$) (fan $\not\perp$ spine if $j_\perp\neq 0$)
		$=0$, ⇒2-D null	Anti-parallel lines with null plane ($j_\perp=0$) Planes of parabolae with null line ($j_\perp\neq 0$)
>0 $(j^2_{thresh}-j^2_\parallel<0)$	no⇒non-potential	$\neq 0$, ⇒3-D null (<0 positive, >0 negative)	Spiral null (fan $\perp$ spine if $j_\perp=0$) (fan $\not\perp$ spine if $j_\perp\neq 0$)
		$=0$, ⇒2-D null	Continuous concentric ellipses ($\perp$ to null line if $j_\perp=0$) ($\not\perp$ to null line if $j_\perp\neq 0$)

$$\mathbf{x}_2'=\begin{pmatrix} \dfrac{(p+2)\sqrt{j_\parallel^2-j_{thresh}^2}}{j_\perp(\sqrt{j_{thresh}^2-(p-1)^2}+j_\parallel)} \\ \dfrac{\sqrt{j_\parallel^2-j_{thresh}^2}}{2j_\perp} \\ 0 \end{pmatrix},$$

$$\mathbf{x}_3=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

with $p>-1$. So, not surprisingly, the fan is not perpendicular to the spine [Fig. 8(c)], but lies in the plane

$$2j_\perp(\sqrt{j_{thresh}^2-(p-1)^2}+j_\parallel)x-4j_\perp(p+2)y+(9(p+1)^2-j_{thresh}^2+j_\parallel^2)z=0\ .$$

When $p=-1$ the field reduces to a two-dimensional null with elliptical field lines in successive planes given by

$$2j_\perp(\sqrt{j_{thresh}^2-4}+j_\parallel)x-4j_\perp y+(j_\parallel^2-j_{thresh}^2)z=0\ ,$$

which are inclined at an angle to the null line along the z-axis [Fig. 8(f)].

VI. CONCLUSION

In this paper we have analyzed the local structure about a linear three-dimensional null. The type of field configurations are found to depend on four parameters. To comprehensively study the localized field about a three-dimensional null, we reduced the 3×3 matrix $\mathbf{M}$ which determines the field to its simplest form by considering the physical character of the field as opposed to reducing the matrix from a mathematical view point. This enabled extra cases to be discovered which were overlooked by previous authors who tackled the problem from the mathematical angle.

In the potential situation one parameter p determines all the possible configurations which are either proper or improper radial nulls and have their spine perpendicular to the fan. It is also possible that for particular values of p the null may reduce to a two-dimensional configuration containing successive X-points in parallel planes.

In a non-potential situation up to four parameters are needed to define all the possible linear structures of a three-dimensional null. It has been found that the relative size of the current parallel to the spine with respect to the threshold current determines whether the null has improper radial, critical spiral or spiral field lines. The field lines in the fan may lie predominantly along one line, known as the major axis of the fan. This axis is parallel to the vector associated with the eigenvalue whose real part has the greatest magnitude out of the two which have real parts of the same sign. We find that the component of current perpendicular to the spine determines the inclination of the fan plane to the spine. The fan does not necessarily tilt about the line of the perpendicular component of current but in general tilts about a different line in the xy-plane.

Obviously the parallel and perpendicular components of current are very important for determining the structure of the null, however, to calculate them the eigenvalues and associated vectors must be found. This, in general, is not simple. On the other hand, j_{thresh} is easy to calculate. It is equal to $\sqrt{b^3/27+c^2/4}$ where $\lambda^3+b\lambda+c=|\lambda\mathbf{I}-\mathbf{S}|=0$ and **S** is the symmetric part of **M**. From all this information the exact structure of the null can be calculated as can the critical value of the parallel component of current that will deform the null from one type to another. There is, however, a relatively quick and easy way of discovering the basic structure of the null without solving the characteristic equation. That is by ascertaining whether **M** is symmetric or not then by finding out what the sign of the determinant of **M** and the sign of the discriminant of the characteristic equation of **M** is. With these three facts you may determine whether the null is two- or three-dimensional, whether it is positive or negative, a spiral or improper radial null and also whether it is potential or non-potential (see Table II for details).

Finally, in our analysis we have considered the linearization of the three-dimensional neutral point purely from the point of view of a Cartesian geometry. Instead we could have linearized with respect to cylindrical or spherical coordinates, however, this approach does not lead to any new forms for the topology of the null. This is because discontinuities are introduced along $x=y=0$ in both cylindrical and spherical polars in either the magnetic field or the current. These are unphysical since we have assumed our field and current are continuous in linearizing about the null.

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