On the nature of three-dimensional magnetic reconnection

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[1] Three-dimensional magnetohydrodynamic reconnection in a finite diffusion region is completely different in many respects from two-dimensional reconnection at an X-point. In two dimensions a magnetic flux velocity can always be defined: two flux tubes can break at a single point and rejoin to form two new flux tubes. In three dimensions we demonstrate that a flux tube velocity does not generally exist. The magnetic field lines continually change their connections throughout the diffusion region rather than just at one point. The effect of reconnection on two flux tubes is generally to split them into four flux tubes rather than to rejoin them perfectly. During the process of reconnection each of the four parts flips rapidly in a virtual flow that differs from the plasma velocity in the ideal region beyond the diffusion region. *INDEX TERMS:* 7835 Space Plasma Physics: Magnetic reconnection; 7827 Space Plasma Physics: Kinetic and MHD theory; 7524 Solar Physics, Astrophysics, and Astronomy: Magnetic fields; 0654 Electromagnetics: Plasmas; *KEYWORDS:* magnetic reconnection, magnetohydrodynamics, magnetic flux

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1. Introduction

[2] Two-dimensional reconnection has been studied for many years and is now fairly well understood, but the way reconnection operates in three dimensions has only been studied relatively recently (see, e.g., *Priest and Forbes* [2000] for a review). General considerations have been given by *Schindler et al.* [1988], who categorized different kinds of reconnection and put forward the notion of general magnetic reconnection. In a companion paper, *Hesse and Schindler* [1988] analyzed the process in terms of Euler potentials. More recently, *Hornig* [2001] and *Hornig and Rastätter* [1998] have proposed an elegant covariant formulation of three-dimensional reconnection.

[3] Several numerical experiments have added to our understanding of three-dimensional (3-D) reconnection. For example, *Lau and Finn* [1996] followed the evolution of a pair of twisted flux tubes. *Dahlburg et al.* [1997] and *Linton et al.* [2000] found that two isolated flux tubes could either bounce, merge, reconnect normally, or tunnel, depending on their twist and inclination. Also, *Galsgaard and Nordlund* [1997] found that an initial configuration including eight null points reconnected by separator reconnection.

[4] Since very little is known about the fundamental nature of 3-D reconnection and we are still in an exploratory phase, we shall focus on the implications of the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}) \quad (1)$$

alone, and so, by analogy with the early stages of dynamo theory, we adopt a kinematic approach and ignore the implications of the equation of motion. Equation (1) arises by taking the curl of Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma}, \quad (2)$$

where $\mathbf{j} = \nabla \times \mathbf{B}/\mu$ and $\eta = (\mu\sigma)^{-1}$ may depend on position. It describes how the magnetic field changes in time owing to two effects on the right-hand side of equation (1), namely, the advection of the magnetic field with the plasma and the diffusion of the magnetic field through the plasma. In most of the universe the magnetic Reynolds number $\left(R_m = \frac{v_0 L_0}{\eta}\right)$ is much larger than unity and so the first term on the right-hand side of equation (1) dominates, the magnetic field is frozen to the plasma, and Ohm's law reduces to

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0}.$$

The exception is in current concentrations where the electric current ($\mathbf{j} = \nabla \times \mathbf{B}/\mu$) and magnetic gradient

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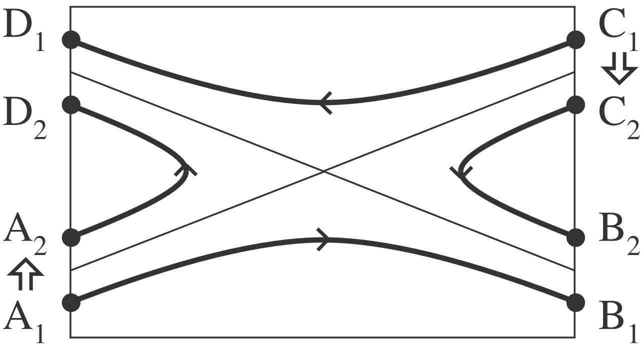

Figure 1. The mapping of field lines in two dimensions (2-D).

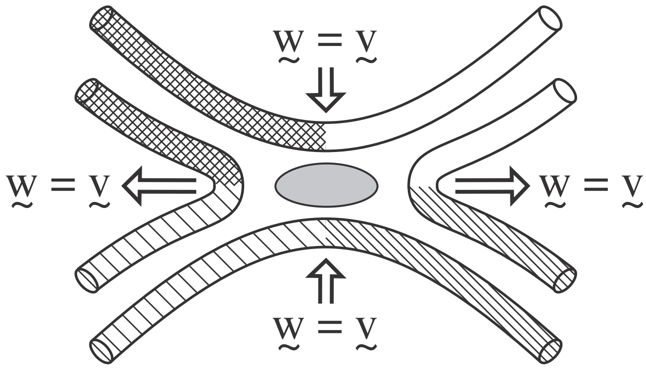

Figure 2. Breaking and rejoining of two flux tubes in 2-D to form two new flux tubes.

become very large, namely at locations where reconnection is occurring.

[5] For a given magnetic field $\mathbf{B}(x, y, z, t)$ the plasma velocity ($\mathbf{v}$) satisfies the induction equation (1) or equivalently Ohm's law (equation (2)) together with Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (3)$$

By comparison, in general we can describe the motion of magnetic flux provided a flux-conserving velocity ($\mathbf{w}$) exists, i.e., provided there exists a $\mathbf{w}$ such that

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{w} \times \mathbf{B}). \quad (4)$$

[6] Here we ask: what is the basic nature of 3-D reconnection in the absence of null points and how does it differ from 2-D reconnection? Section 2 summarizes the main features of 2-D reconnection, while section 3 shows that none of these features are present in 3-D reconnection. For example, as proved in section 3, for 3-D reconnection with a finite diffusion region, in general a flux tube velocity does not exist. Finally, an example is described of the way reconnection occurs in a simple magnetic configuration.

2. The Basic Properties of 2-D Reconnection

[7] In two dimensions, magnetic reconnection possesses several fundamental properties, which we shall enumerate below. Many of these have previously been implicit rather than being explicitly stated and have been assumed to be an intrinsic part of the nature of reconnection, but wrestling with the fundamentals of 3-D reconnection has forced us to examine whether indeed they are essential properties of reconnection as a whole or just of 2-D reconnection.

[8] 1. A differentiable flux tube velocity ($\mathbf{w}$) satisfying equation (4) always exists everywhere except at null points. This velocity has a hyperbolic singularity at an X-type null point of $\mathbf{B}$, where the reconnection takes place. The magnetic flux moves at the velocity $\mathbf{w}$ and slips through the plasma, which itself moves at $\mathbf{v}$.

[9] 2. The mapping of field lines in 2-D is discontinuous. Consider the field of an X-type neutral point inside a rectangular boundary, say. A footpoint (A_1) on the left-hand boundary below the separatrix in Figure 1 maps along the magnetic field to a point B_1 on the right-hand boundary. As one moves, however, across the separatrix from A_1 to A_2, there is a sudden jump in the mapping point from the right-hand boundary to a point D_2 on the left-hand boundary. At the same time there is a discontinuity in the mapping of points D_1 to B_2. This mapping discontinuity is associated with the fact that field lines break at one point.

[10] 3. While they are in the diffusion region, field lines preserve their connections. The exception is the X-point, where the field lines break and their connections are changed.

[11] 4. Reconnecting flux tubes rejoin perfectly (Figure 2). That is, to any tube that is going to reconnect, there exists a counterpart with which it will rejoin perfectly. The net effect is that two tubes move towards the diffusion region (at velocity $\mathbf{w} = \mathbf{v}$), they break, and they rejoin perfectly to form two new flux tubes that move out (at $\mathbf{w} = \mathbf{v}$).

[12] 5. When a flux tube is partly in a diffusion region (D), $\mathbf{w} = \mathbf{v}$ on both parts of the tube that are outside, while $\mathbf{w} \neq \mathbf{v}$ on the segment that lies in the diffusion region. Thus the two wings of the flux tube (outside D) in Figure 3 are moving with the plasma, while the central segment (in D) is slipping through the plasma.

[13] As an example, consider a steady-state magnetic field of an X-point with uniform current having components

$$B_x = y, \quad B_y = a\,x$$

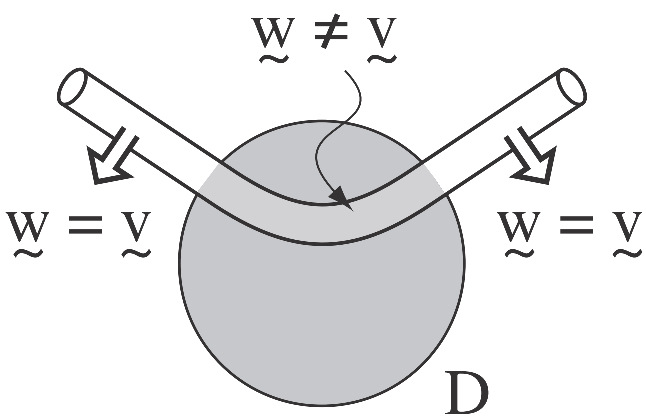

Figure 3. The behavior of a flux tube in 2-D when partly in a diffusion region.

where a is a constant. The component $\mathbf{w}_\perp$ of the flux tube velocity ($\mathbf{w}$) perpendicular to $\mathbf{B}$ is determined by

$$\mathbf{E} + \mathbf{w} \times \mathbf{B} = \mathbf{0}$$

(with $\mathbf{E} = E_0\hat{\mathbf{z}}$ constant) to be

$$\mathbf{w}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

and so to have components

$$w_x = \frac{-E_0\ a\ x}{a^2x^2 + y^2}, \quad w_y = \frac{E_0\ y}{a^2x^2 + y^2}.$$

(The component of $\mathbf{w}$ parallel to the magnetic field is arbitrary in this analysis and could be determined by an additional constraint such as incompressibility.) It can be seen that $\mathbf{w}$ has a point singularity at the X-point $(x, y) = (0, 0)$. Indeed, a signature of two-dimensional reconnection is that the inflow speed (w_y) of magnetic flux behaves like y^{-1} along the inflow (the y-axis), while the outflow speed (w_x) behaves like $-x^{-1}$ along the outflow (the x-axis).

[14] Once these basic properties have been assumed and established, even implicitly, it is then possible to build models for different regimes of reconnection and, for instance, to calculate the reconnection rates [e.g., *Priest and Forbes*, 2000]. The question then arises: does the same description in terms of flux tube velocities work in three dimensions?

3. The Properties of 3-D Reconnection

[15] In three dimensions, surprisingly, none of the above properties carry over and so the nature of reconnection is profoundly different from our accustomed way of thinking in two dimensions.

[16] 1. A flux tube velocity ($\mathbf{w}$) does not generally exist in MHD, as proved in the following theorem (although nonMHD effects such as double layers could possibly negate the theorem).

[17] Theorem: for an isolated 3-D magnetohydrodynamic diffusion region, a flux-conserving flow ($\mathbf{w}$) does not exist in general.

[18] Proof: consider a finite diffusion region (D). Suppose $\mathbf{w}$ does exist. Then we have

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{w} \times \mathbf{B}). \tag{5}$$

Using Faraday's law (equation (3)), we can uncurl this equation to obtain:

$$\mathbf{E} + \mathbf{w} \times \mathbf{B} = \nabla F, \tag{6}$$

where F is a scalar potential.

[19] Outside the diffusion region (D), $\mathbf{w} = \mathbf{v}$ and $\nabla F = 0$, and so without loss of generality we may assume $F = 0$. However, taking the dot product of equation (6) with $\mathbf{B}$ makes the second term on the left of equation (6) vanish, so that

$$\mathbf{B} \cdot \nabla F = \mathbf{B} \cdot \mathbf{E},$$

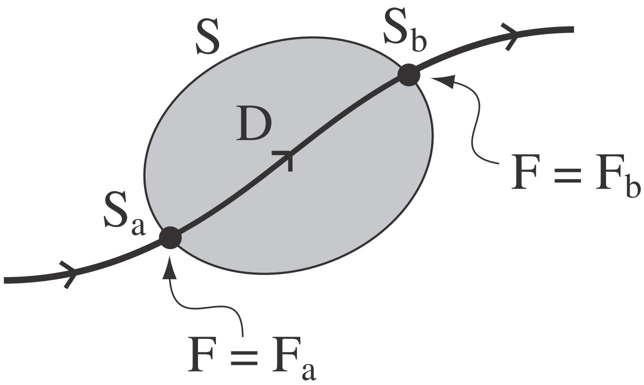

Figure 4. The notation for the potential (F) on a field line that passes through the surface (S) of a diffusion region (D) in three dimensions (3-D).

which may be integrated to give

$$F = \int E_\parallel\ ds, \tag{7}$$

where the integration is along a field line and $E_\parallel$ is the component of $\mathbf{E}$ parallel to $\mathbf{B}$.

[20] Split the surface (S) of the diffusion region into two parts, on one (S_a) of which the magnetic field lines are entering the diffusion region and on the other (S_b) of which the field lines are leaving D. Suppose $F = F_a = 0$ on field lines before they enter D through S_a and $F = F_b$ at the points where they leave through S_b (Figure 4). Then, in general, equation (7) implies that F_b is nonzero and so F does not vanish on field lines beyond S_b, i.e., outside D. However, this is a contradiction and so we conclude that $\mathbf{w}$ does not in general exist.

[21] 2. The mapping of field lines is continuous. In general for regions in which there is no 3-D null point the mapping of field lines from one part of the boundary of a region to another is continuous. Consider for example the field $(B_x, B_y, B_z) = (y, x, 1)$ inside a cube, formed by adding a uniform field in the z-direction to a 2-D X-point (Figure 5). In this case a point A_1 on the bottom of the cube will map to a point B_1 on the top. This time, as the location of the initial point moves slowly from A_1 to A_2, the final point moves rapidly but continuously along the top of the box from B_1 to D_2. At the same time, the footpoint C_1 maps to D_1 while C_2 maps to B_2. In other words, the mapping is now continuous but, as the initial point moves slowly across the remnant of the separatrix that existed in 2-D (called a quasi-separatrix in 3-D) [*Priest and Démoulin*, 1995], the end-point of the field line performs a rapid flipping motion [*Priest and Forbes*, 1992].

[22] 3. While they are in a 3-D diffusion region, magnetic field lines continually change their connections.

[23] 4. Two flux tubes don't generally break and reform perfectly in 3-D to give two flux tubes. Instead, each half of the original tubes joins to a different half, as indicated in Figure 6.

[24] 5. When the two flux tubes are partly in the diffusion region and so are in the process of reconnecting, they split into four parts, each of which flips in a different manner. In

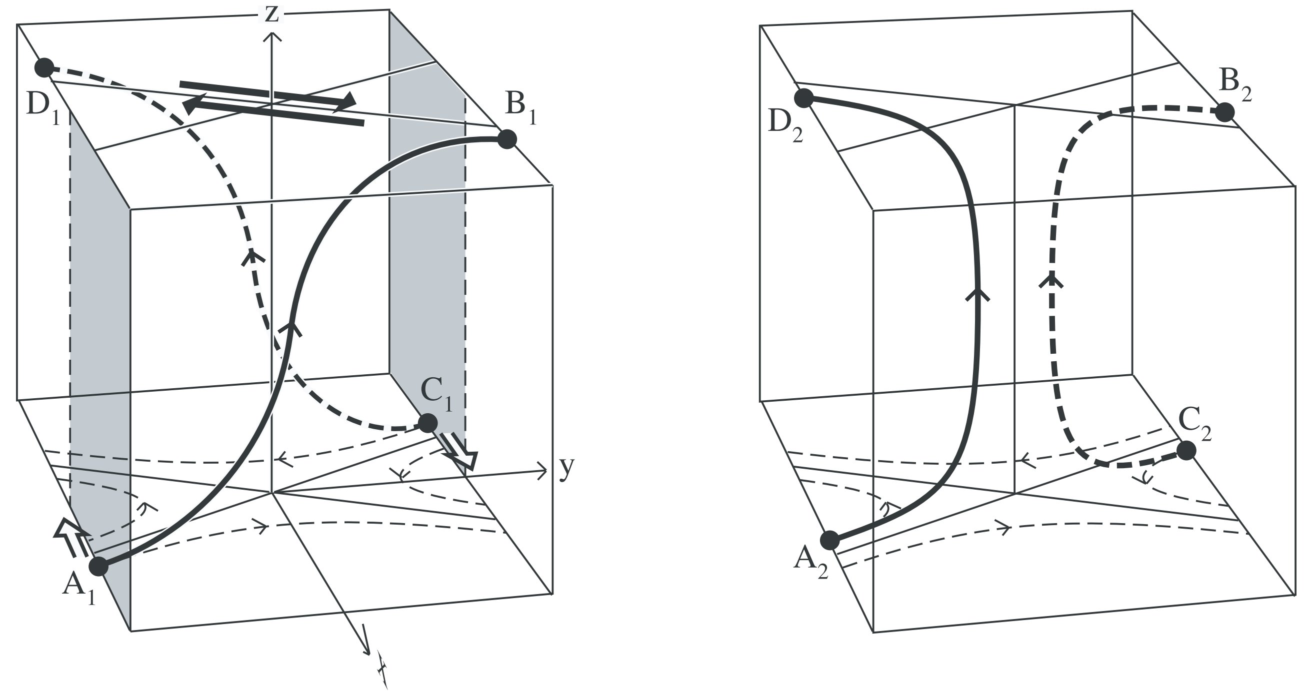

Figure 5. The mapping of field lines in 3-D.

particular, if we follow a flux tube through the diffusion region D we find that beyond D it moves with a velocity different from the plasma velocity. This is a manifestation of the nonexistence of a flux conserving velocity **w**.

[25] Thus we can define a flux tube velocity $\mathbf{w}_A$ by picking points on the section A (which moves with the plasma) and tracing field lines from them across the diffusion region and repeating the procedure at later times. Suppose that initially, when the flux tube lies completely outside the diffusion region, the section A maps to a section B, and repeat the procedure for starting points on the section B to define a flux tube velocity $\mathbf{w}_B$. In a similar way, define flux tube velocities $\mathbf{w}_C$ and $\mathbf{w}_D$ from field lines that start from points on sections C and D. While the flux tubes lie outside the diffusion region, $\mathbf{w}_A$ is identical to $\mathbf{w}_B$ and $\mathbf{w}_C$ is identical to $\mathbf{w}_D$. As soon as the flux tubes enter the diffusion region they start reconnecting and the sections A and C no longer map to B and D, respectively, while $\mathbf{w}_A$ and $\mathbf{w}_C$ are no longer equal to $\mathbf{w}_B$ and $\mathbf{w}_D$, respectively, because of the nonexistence of a single flux tube velocity **w**. Furthermore, after reconnection has been completed, usually the sections A and C do not map to D and B, respectively, unlike the two-dimensional case.

[26] Note that the condition that the diffusion region be compact is an assumption, but it is in our view a reasonable one. In our proof, the current itself does not need to be bounded, rather it is $\nabla \times (\eta \mathbf{j})$ that is assumed bounded.

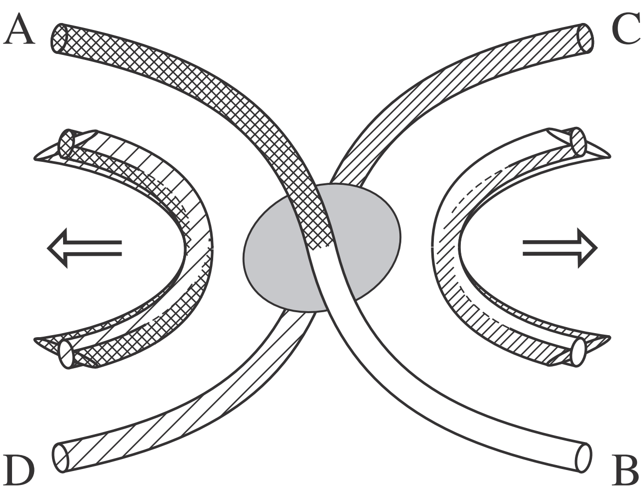

Figure 6. Breaking and partial rejoining of two flux tubes in 3-D to form four new flux tubes.

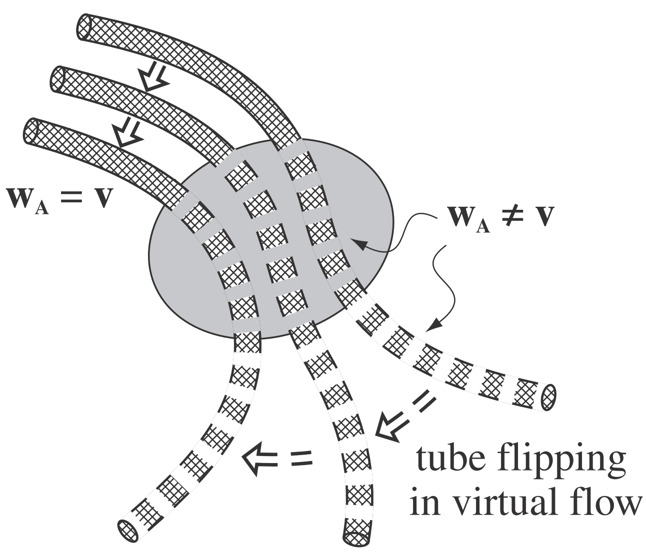

Figure 7. The flipping during reconnection of one of the four parts (A, say) of the initial two tubes in Figure 6.

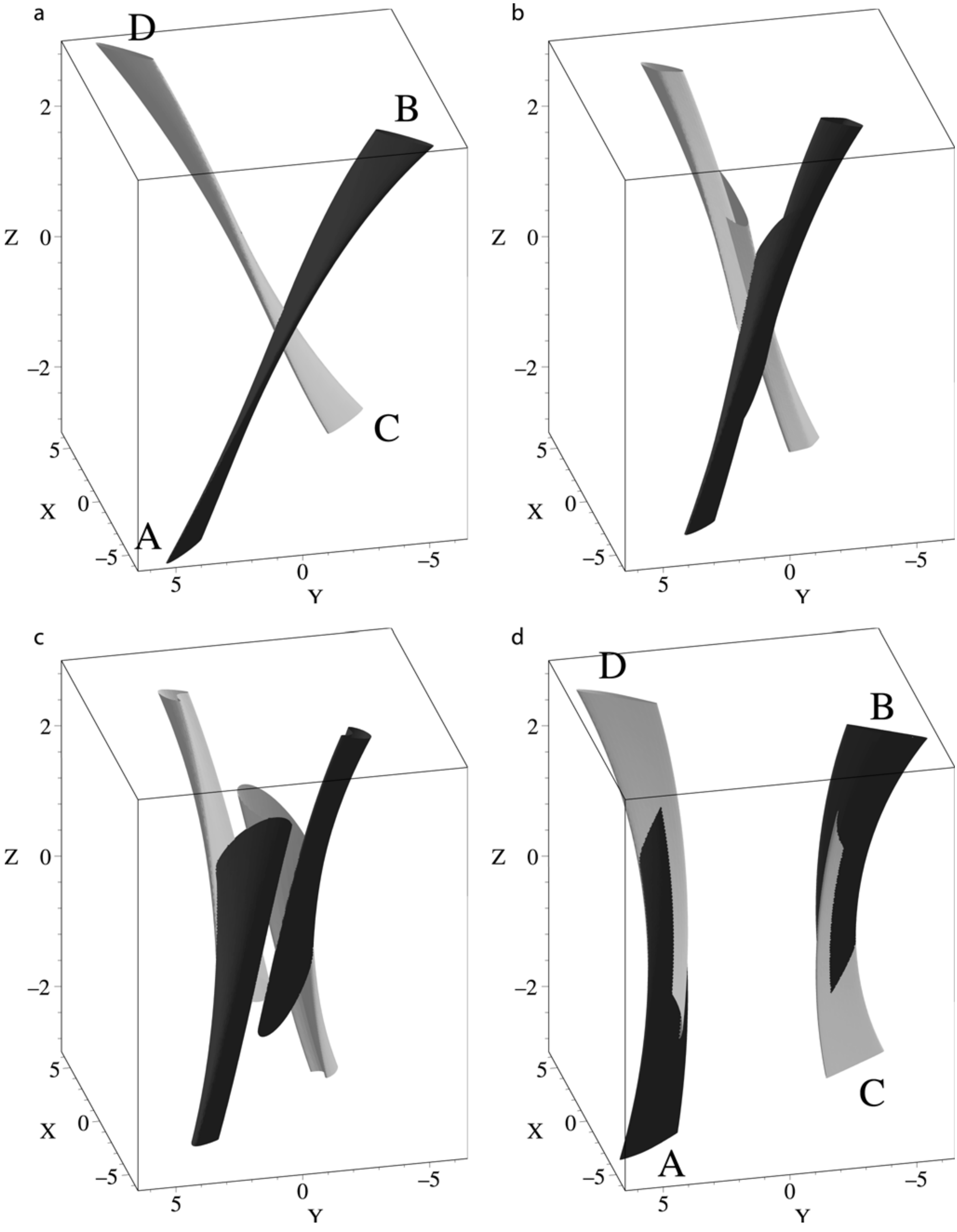

Figure 8. Kinematic reconnection of two flux tubes (integrated from the cross sections A, B, C and D) for the magnetic field $\mathbf{B} = (y, k^2x, b_0)$. (a) The initial two tubes. (b) The tubes start to split and (c) the tubes flip past each other. (d) The final four nonmatching flux tubes once they have left the diffusion region.

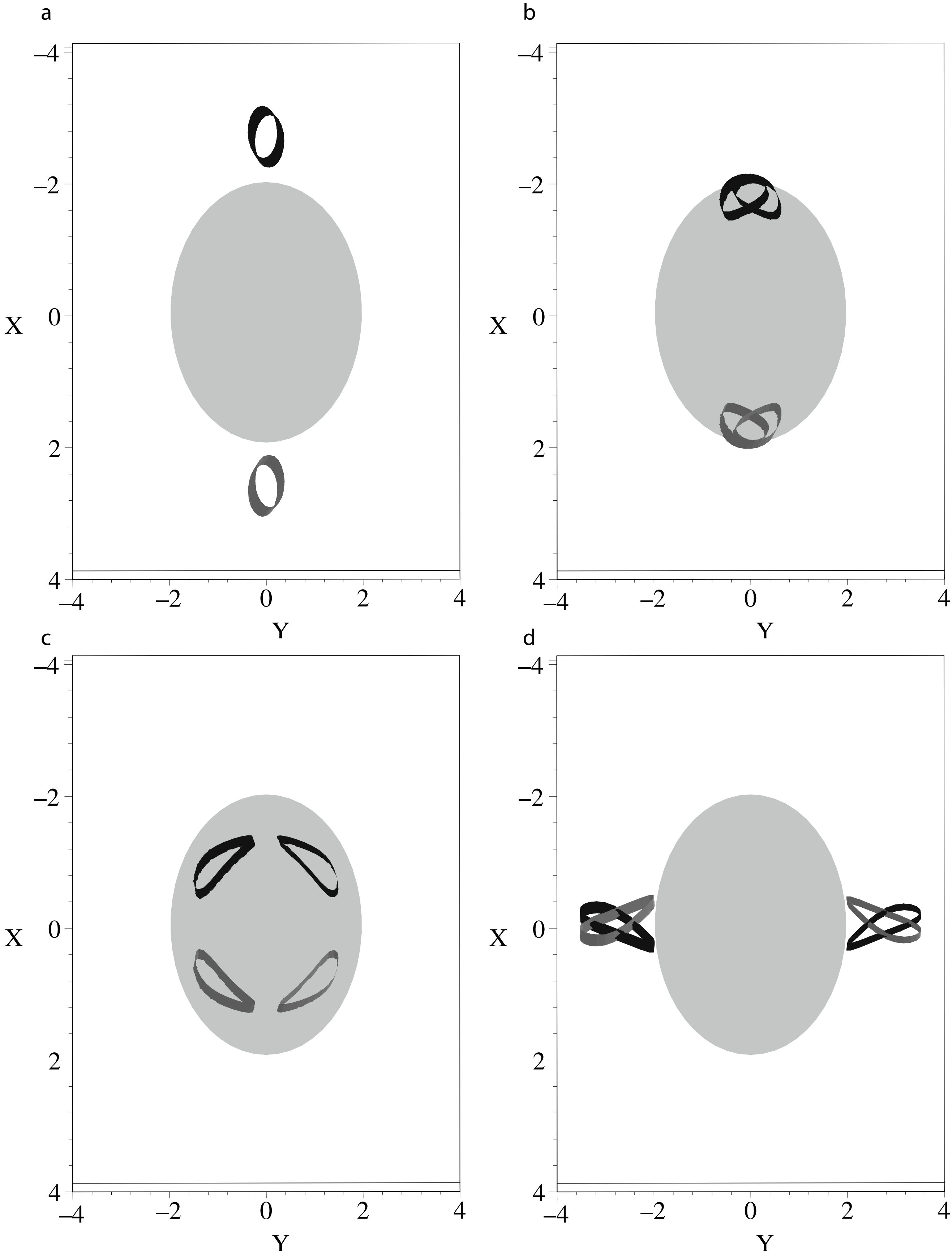

Figure 9. Cross sections in the $z = 0$ plane of the four tubes shown in Figure 8. The shaded region shows the diffusion region. (a) The two initial tubes, and (b and c) the tubes split and flip past each other. (d) The nonmatching of the final tubes.

Thus if the resistivity were uniform the current would need to be localized, whereas if the current were uniform the resistivity would be localized. In a realistic plasma there would tend to be a combination of both effects: for example, a classical Spitzer resistivity depends on temperature and so tends to be higher in a current concentration.

4. Example of 3-D Reconnection in a Simple Configuration

[27] The properties discussed above are illustrated in the following example, which has a 2.5-dimensional magnetic field but a fully three-dimensional diffusion region (D) and fully three-dimensional flux tube velocities $\mathbf{w}_A$, $\mathbf{w}_B$, $\mathbf{w}_C$, $\mathbf{w}_D$ obtained by integrating from sections A, B, C, D, as in Figure 6. As discussed previously, we solve here only the induction equation (1), so this example is a kinematic solution of the MHD equations. We start by specifying the magnetic field

$$\mathbf{B} = \left(y, k^2x, b_0\right), \tag{8}$$

where k and b_0 are constants. This is simply a generalization of the field shown in Figure 5, which possesses a uniform current $\mathbf{j} = ((k - 1)/\mu)\ \hat{\mathbf{z}}$. The parallel electric field ($E_{\|}$) is now chosen to be localized so that, in turn, η is localized within a three-dimensional region D, approximately defined by $-2 \leq x, y, z \leq 2$. The full electric field (**E**) and the plasma velocity (**v**) are then finally calculated. The field (equation (8)) has also been considered independently by *Titov et al.* [2002, 2003], who have analyzed kinematic and force-free pinching of a so-called "hyperbolic flux tube."

[28] We illustrate the process as follows: two symmetric (elliptical) flux tube cross-sections are initially chosen, centered on ($\pm x_1$, 0, 0), where $x_1 \gg 2$. For each of these cross-sections we now integrate an equal distance in either direction along the magnetic field to find two new cross-sections, either of which also defines the corresponding initial flux tube. Integrating along field lines passing through the boundaries of the resulting four cross-sections gives two initial symmetric flux tubes in the ideal region, on either side of D. The four cross-sections are chosen such that they never pass through D but are moved always with the ideal plasma velocity **v**, which carries them across the quasi-separatrices of **B**. We follow the motion of the flux tubes by integrating the field lines through the four cross-sections (A, B, C, D, in Figure 8) at each time. As long as the flux tubes remain outside the diffusion region, the tubes integrated from cross sections A and B (or C and D) coincide (see Figure 8a).

[29] Once the tubes reach the diffusion region, the description becomes less straightforward. The tubes integrated from cross sections A and B (or C and D) do not coincide anymore. This is due to the nonexistence of a common flux-conserving velocity **w**. Note that we stop the integration of the flux tubes at a certain length, for clarity of the image. As soon as the tubes enter D, they begin to slip through the plasma, but at different rates, so that the identity of the initial two tubes is lost and they each split into two separate tubes, as shown in Figure 8b. The four tube-halves then flip past each other (Figure 8c). Since the parts of the tubes beyond the diffusion region do not move at the ideal velocity (**v**) they continuously change their connections with the ambient field as sketched in Figure 7. As discussed before, we can think of the tubes as flipping in a "virtual" flow. For example, the tube defined by integrating from cross-section A moves with a flow $\mathbf{w}_A$ (Figure 7), which differs from the plasma velocity both in and beyond the diffusion region, whereas tubes from sections B, C, D move with velocities $\mathbf{w}_B$, $\mathbf{w}_C$, $\mathbf{w}_D$, respectively. When the tubes reemerge from D (Figure 8d), we can see that the configuration of two unique tubes is not recovered, as none of the final four tubes are coincident. This nonuniqueness of the final four tubes is even more apparent when we just look at the evolution of the cross-sections of the four tubes in the z = 0 plane (see Figure 9). Although each pair of tube sections intersects at four points, they have certainly not fully reconnected.

5. Conclusions

[30] The question we have addressed here is to what extent out natural understanding of reconnection in 2-D carries over into 3-D. The idea of a flux velocity is a natural extension of an ideal plasma velocity. It is therefore important to determine whether such a velocity exists, i.e., whether the magnetic field can be considered as transported by any flow, if we want to know whether the topology of the magnetic field has changed or not. If it does exist then there is no topology change and no reconnection. If it does not exist, then the topology changes.

[31] In three dimensions the magnetic helicity is in general not conserved when $\mathbf{E} \cdot \mathbf{B} \neq 0$. The mismatching implies magnetic helicity nonconservation, but the reverse is not true since there are examples where $\mathbf{E} \cdot \mathbf{B} \neq 0$ and the tubes match up perfectly, as we shall describe in a future paper.

[32] We have proved that in a region of nonvanishing magnetic field a localized nonideal term in the induction equation will in general lead to the nonexistence of a flux conserving flow for the evolution of the magnetic field. The result is that two flux tubes split and flip when they enter the nonideal region, as demonstrated in an explicit example. In addition, we have given a simple example in which, even after the reconnection process is completed, the flux tubes do not join up again. This implies that the process of three-dimensional reconnection in a region of nonvanishing magnetic field is much more involved than we would expect from two-dimensional models. We shall aim to study the process in more detail, both analytically and numerically, in future.

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