Plasma beta limits for magnetic annihilation models

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Magnetic annihilation occurs when two oppositely directed magnetic fields are brought together by a plasma flow. Several exact nonlinear solutions exist which typically depend on the ratios of plasma pressure to magnetic pressure (the plasma beta), inflow speed to global Alfvén speed (the Alfvén Mach number) and of the advective to diffusive terms of the induction equation (the Lundquist number). Ensuring that the plasma pressure is everywhere positive restricts the freedom of choice of these parameters, however. Restrictions on the plasma beta are derived for the cases of two- and three-dimensional annihilation and two-dimensional reconnective annihilation. At the inflow speeds typically required for fast reconnection in diffuse astrophysical plasmas the minimum plasma beta is several orders of magnitude larger than the observed values of unity or less. In other words, at the observed plasma beta the models are only valid for extremely small annihilation rates.

I. INTRODUCTION

Many models for magnetic annihilation and reconnection in laboratory, space, solar and astrophysical plasmas have now been proposed for the conversion of magnetic energy into other forms and the changing of magnetic connectivity and topology. Such processes are usually invoked in plasmas with a low plasma beta (ratio of plasma pressure to magnetic pressure), so that the magnetic pressure dominates the plasma pressure and the main energy source is magnetic. However, Priest (Ref. 1) has recently realized that the simplest model of two-dimensional (2D) stagnation-point flow bringing together oppositely directed magnetic field lines and annihilating them only works when the value of the plasma pressure at an external reference point is sufficiently large. In practice this implies that the model is only valid at very low annihilation rates. Priest's analysis was only a simple order of magnitude one, so here we repeat it more precisely (Sec. II) and also extend it to the cases of three-dimensional annihilation (Sec. III) and reconnective annihilation (Sec. IV).

II. TWO-DIMENSIONAL STAGNATION FLOW

Sonnerup and Priest (Ref. 2) considered the simple case of magnetic annihilation in a two-dimensional Cartesian domain containing a narrow diffusion region, within which ideal magnetohydrodynamics (MHD) breaks down due to the presence of large magnetic gradients. The region (Fig. 1) is bounded by $-L_e \leqslant x \leqslant L_e$ and has an external magnetic field, $B_e\hat{\mathbf{y}}$, and pressure, p_e, at $x=L_e$. The density, ρ_e, is taken to be uniform. Oppositely directed magnetic fields,

$$\mathbf{B}=B(x)\hat{\mathbf{y}},$$

with $B(x)$ an odd function of x, are driven together by a stagnation flow,

$$\mathbf{v}=\frac{v_e}{L_e}(-x\hat{\mathbf{x}}+y\hat{\mathbf{y}}),$$

v_e being the speed at $x=L_e, y=0$. This is defined on the region $-L_e \leqslant x \leqslant L_e, -L_e \leqslant y \leqslant L_e$ and causes the magnetic field to annihilate at $x=0$.

The resulting z-component of Ohm's law in a two-dimensional steady state is

$$E-\frac{v_e}{L_e}xB=\eta\frac{dB}{dx},$$

where the electric field, $\mathbf{E}=E\hat{\mathbf{z}}$, is uniform and constant and η is the magnetic diffusivity. Writing $x'=x/L_e$, $y'=y/L_e$, $B'=B/B_e$, $E'=E/(v_eB_e)$ and the global magnetic Reynolds number as

$$R_m=\frac{v_eL_e}{\eta},$$

this becomes

$$\frac{dB'}{dx'}+R_mx'B'-R_mE'=0. \tag{1}$$

The boundary conditions are such that the magnetic field vanishes at $x'=0$ and matches the external field $(B_e\hat{\mathbf{y}})$ at $x'=1$. From now on the primes will be dropped, the use of dimensionless equations being indicated by the presence of dimensionless parameters such as the magnetic Reynolds number.

The solution to Eq. (1) depends only on R_m

$$B(x)=\exp\left(\frac{R_m(1-x^2)}{2}\right)\frac{\int_0^x \exp(R_my^2/2)dy}{\int_0^1 \exp(R_my^2/2)\ dy},$$

and can also be written in terms of Kummer functions as

$$B(x)=x\frac{M(1,\frac{3}{2},-\frac{1}{2}R_mx^2)}{M(1,\frac{3}{2},-\frac{1}{2}R_m)}. \tag{2}$$

The value of the constant electric field is obtained from the boundary condition $B=B_e$ at $x=L_e$ and is

$$E=\frac{\exp(R_m/2)}{R_m\int_0^1\exp(R_my^2/2)dy}=\left(R_mM\left(1,\frac{3}{2},-\frac{1}{2}R_m\right)\right)^{-1}. \tag{3}$$

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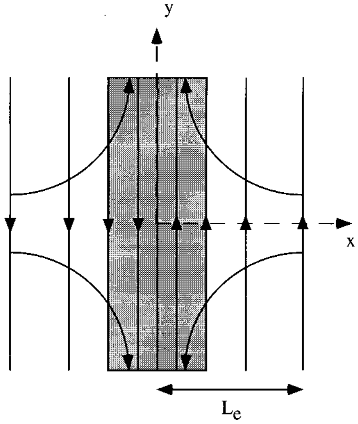


FIG. 1. Annihilation of a one-dimensional magnetic field (vertical lines) by a two-dimensional stagnation flow (curved lines). The shaded box is the diffusion region.

$M(a,b,\zeta)$ is the confluent hypergeometric Kummer function, which satisfies Kummer's equation, $\zeta M''+(b-\zeta)M'-aM=0$, the primes denoting derivatives with respect to ζ. The magnetic field solution of Eq. (2) is just that of Anderson and Priest (Ref. 3) re-written using the identity

$$M(a,b,\zeta)=\exp(\zeta)M(b-a,b,-\zeta).$$

As well as satisfying the equations of induction and mass continuity $(\nabla\cdot(\rho_e\mathbf{v})=0)$ the above solution also satisfies the equation of motion and so is one of the few exact solutions of the MHD equations. The equation of motion simply determines the plasma pressure (in dimensionless form $p'=2\mu_0 p/B_e^2$) as

$$p'=\beta_e+M_e^2(1-x'^2-y'^2)+(1-B'^2), \tag{4}$$

where the plasma beta is

$$\beta_e=\frac{2\mu_0 p_e}{B_e^2},$$

μ_0 being the permeability of free space. The annihilation rate for this situation may be measured by the Alfvén Mach number,

$$M_e=\frac{v_e}{v_a},$$

where $v_a=B_e/(\mu_0\rho_e)^{1/2}$ is the Alfvén speed at the edge of the region. Priest (Ref. 1) has used order of magnitude arguments to deduce a limitation on the annihilation rate, namely

$$M_e<\frac{\beta_e}{S_e}, \tag{5}$$

where S_e is the Lundquist number, assumed to be much larger than unity and defined by

$$S_e=\frac{v_a L_e}{\eta}.$$

Note that the magnetic Reynolds number, Lundquist number and Alfvén Mach number are related by $R_m=M_e S_e$. Equation (5) may be re-written as a limitation on β_e, namely

$$\beta_e>M_e S_e. \tag{6}$$

In particular, in order to attain reconnection faster than the Sweet-Parker rate ($M_e=S_e^{-1/2}$) one needs an enormous plasma beta, $\beta_e>S_e^{1/2}$.

The argument used to obtain these inequalities is based on the behavior of the magnetic field, Eq. (2). For ζ large and negative the Kummer function may be written as the following asymptotic expansion (cf. Ref. 4):

$$M(a,b,\zeta)=\frac{\Gamma(b)}{\Gamma(b-a)}(-\zeta)^{-a}\sum_{r=0}^{\infty}\frac{a_r}{r!}(-\zeta)^{-r} + \frac{\Gamma(b)}{\Gamma(a)}\cos[\pi(b-a)]\zeta^{a-b}\exp(\zeta)\sum_{r=0}^{\infty}\frac{b_r}{r!}\zeta^{-r}, \tag{7}$$

where $\Gamma(\zeta)$ is the gamma function, $a_0=b_0=1$, $a_r=\Pi_{j=0}^{r-1}(a+j)(j+a-b+1)$ and $b_r=\Pi_{j=0}^{r-1}(b-a+j)\times(j-a+1)$ for $r\geqslant 1$. For $a=1$, $b=3/2$ the coefficient of the second series is zero so, to leading order,

$$M\left(1,\frac{3}{2},\zeta\right)\simeq-\frac{1}{2\zeta}.$$

Thus, for large R_m, the denominator of Eq. (2) is R_m^{-1} to leading order. The peak in the magnetic field strength occurs when x is of order $R_m^{-1/2}$ where the asymptotic expansion is no longer valid. Instead, the convergent series form of the Kummer function,

$$M(a,b,\zeta)=1+\sum_{n=1}^{\infty}\frac{\zeta^n}{n!}\prod_{j=0}^{n-1}\frac{a+j}{b+j}, \tag{8}$$

may be used. To lowest order for small ζ this gives $M(1,3/2,\zeta)\simeq 1$.

For large R_m and $x<O(R_m^{-1/2})$, the magnetic field may be approximated by

$$B(x)\simeq R_m x. \tag{9}$$

Far away from the peak in the magnetic field, on the other hand, the asymptotic expansion may be used for both numerator and denominator, giving

$$B(x)\simeq\frac{1}{x}. \tag{10}$$

These approximations represent balances between the first and third, and second and third terms in Eq. (1), respectively (with $E'=1$ to leading order in R_m). Priest estimated the location of the maximum magnetic field as being at the point where the two limiting expansions, Eqs. (9) and (10), are equal, namely $x\simeq R_m^{-1/2}$ with $B_{max}\simeq R_m^{1/2}$ He further considered Eq. (4) in the sub-sonic and sub-Alfvénic limit where the second term on the right side is neglected. The minimum pressure then occurs where the magnetic field is maximum, i.e.,

$$p_{min}=\beta_e+1-B_{max}^2. \tag{11}$$

Requiring this minimum pressure to be positive implies that

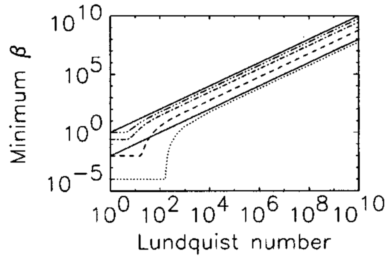


FIG. 2. $\beta_{\min}$, as a function of the Lundquist number, S_e. The broken curves have $M_e=0.01,0.1,0.5$, and 1.0 in ascending order while the solid curves give the order-of-magnitude estimate $\beta_{\min}=M_eS_e$ for the cases $M_e=0.01$ (lower) and 1.0 (upper).

$$M_e<\frac{\beta_e+1}{S_e},$$

which further reduces to Eq. (5) when $R_m\gg 1$. The physical reason for the beta limitation is that in the stagnation flow it is a fall in plasma pressure as the plasma approaches the stagnation point that causes the magnetic field strength to rise; since the magnetic field increases by a large factor ($R_m^{1/2}$), a correspondingly large pressure gradient is required.

A more exact beta limit may now be calculated numerically as follows, using the whole of Eq. (4). It can be seen that the minimum pressure occurs on the planes $y=\pm 1$ where $M_e^2x^2+B^2(x)$ is maximum. Setting the derivative of this expression with respect to x equal to zero, using the identities

$$\frac{\partial}{\partial\zeta}M(a,b,\zeta)=\frac{a}{b}M(a+1,b+1,\zeta),$$

$$a\zeta M(a+1,b+1,\zeta)=b(1-b+\zeta)M(a,b,\zeta)+b(b-1)M(a-1,b-1,\zeta),$$

and letting $\xi=x_*R_m^{1/2}$ leads to the following equation for the minimum pressure location, x_*, in terms of ξ:

$$\xi^2M^2\left(1,\frac{3}{2},-\frac{1}{2}\xi^2\right)-M\left(1,\frac{3}{2},-\frac{1}{2}\xi^2\right)-M_e^2M^2\left(1,\frac{3}{2},-\frac{1}{2}R_m\right)=0.$$

This equation for ξ may be solved numerically for given values of M_e and S_e. The corresponding minimum beta is

$$\beta_{\min}=M_e^2x_*^2+B^2(x_*)-1. \quad (12)$$

The results of this procedure are presented in Fig. 2. The curves start with a constant non-zero value for low Lundquist number because in these cases the magnetic field strength decreases monotonically from B_e at $x=L_e$ to zero at $x=0$ with no stationary point, corresponding to a weak current sheet of width $2L_e$. Equation (12) shows that the minimum plasma beta is then M_e^2 occurring at $x_*=1$. The order-of-magnitude approximation, Eq. (6), is also plotted for each curve for comparison. In all the cases with a local maximum (inclined straight line) the value of ξ varied between 1.3 and 1.5. This latter case corresponds to a current sheet of width $R_m^{-1/2}$.

III. THREE-DIMENSIONAL STAGNATION FLOW

A three-dimensional flow of the form,

$$\mathbf{v}=\frac{v_e}{L_e}(-x\hat{\mathbf{x}}+\alpha_y y\hat{\mathbf{y}}+\alpha_z z\hat{\mathbf{z}}),$$

was also considered (Ref. 2), with $\alpha_y+\alpha_z=1$ to ensure incompressibility and defined for $-L_e\leq x\leq L_e$, $-L_e\leq y\leq L_e$, $-L_e\leq z\leq L_e$. The magnetic field, meanwhile, is generalized to have two components, but is still dependent on only one coordinate

$$\mathbf{B}=B_y(x)\hat{\mathbf{y}}+B_z(x)\hat{\mathbf{z}}.$$

Thus, as the flow carries the field in towards the plane $x=0$ the field lines rotate but remain parallel to that plane. The y- and z-components of the induction equation then give

$$\frac{\eta L_e}{v_e}\frac{d^2B_y}{dx^2}+x\frac{dB_y}{dx}+\alpha_yB_y=0,$$

$$\frac{\eta L_e}{v_e}\frac{d^2B_z}{dx^2}+x\frac{dB_z}{dx}+\alpha_zB_z=0.$$

After putting $x'=x/L_e$, $B_y'=B_y/B_e$ and $B_z'=B_z/B_e$, B_e being the magnetic field strength at $x=L_e$, and letting $\xi=x'R_m^{1/2}$, the two induction equation components can be written in the form

$$\frac{d^2I}{d\xi^2}+\xi\frac{dI}{d\xi}+\alpha I=0. \quad (13)$$

The y-component of the induction equation has $I=B_y$ and $\alpha=\alpha_y$ while the z-component has $I=B_z$ and $\alpha=\alpha_z$.

Sonnerup and Priest solved Eq. (13) in terms of two odd and even series solutions, I_α^e and I_α^o, which they gave explicitly. In fact, these two functions can be written in terms of Kummer functions as

$$I_\alpha^e(\xi)=M\left(\frac{\alpha}{2},\frac{1}{2},-\frac{\xi^2}{2}\right),$$

$$I_\alpha^o(\xi)=\xi M\left(\frac{\alpha+1}{2},\frac{3}{2},-\frac{\xi^2}{2}\right).$$

Sonnerup and Priest expressed their boundary conditions in terms of the y- and z-components of the electric and magnetic fields at $x=0$. For the purposes of this study, however, the boundary conditions are given in terms of the magnetic field components at $x=0$ and $x=1$: $B_y(0)=B_{y0}$, $B_y(1)=B_{ye}$, $B_z(0)=B_{z0}$ and $B_z(1)=B_{ze}$. The magnetic field component in the y-direction is

$$B_y=B_{y0}M\left(\frac{\alpha_y}{2},\frac{1}{2},-\frac{1}{2}\xi^2\right)+\left[B_{ye}-B_{y0}M\left(\frac{\alpha_y}{2},\frac{1}{2},-\frac{1}{2}R_m\right)\right]\times\frac{\xi}{R_m^{1/2}}\frac{M\left(\frac{\alpha_y+1}{2},\frac{3}{2},-\frac{1}{2}\xi^2\right)}{M\left(\frac{\alpha_y+1}{2},\frac{3}{2},-\frac{1}{2}R_m\right)},$$

with a similar expression for B_z being obtained from this by changing the y-subscripts for z's. For large ξ and R_m there is an initial growth by several orders of magnitude in the size

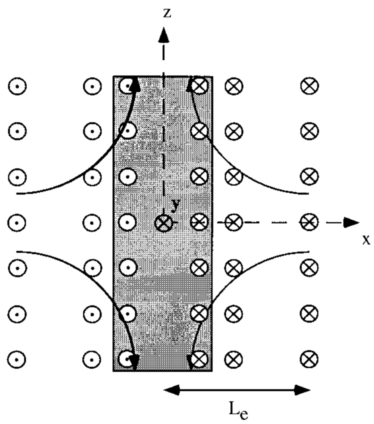


FIG. 3. Annihilation of a one-dimensional magnetic field (into and out of the page) by a two-dimensional stagnation flow (curved lines).

of the initial terms of the series, which is then cancelled by the next few terms. Unless many significant digits are available, loss of precision results from subtracting two numbers whose difference is orders of magnitude smaller than each individual number. In this situation it is preferable to use the asymptotic form of the Kummer function which applies for ζ real and negative, Eq. (7). This is well-defined for $a \neq b$; when $a=b$ the identity $M(a,a,\zeta)=\exp\zeta$ can be used instead. The two series comprising the asymptotic expansion are summed as far as the point at which the terms increase in magnitude.

The momentum equation for this more general case gives the dimensionless pressure as

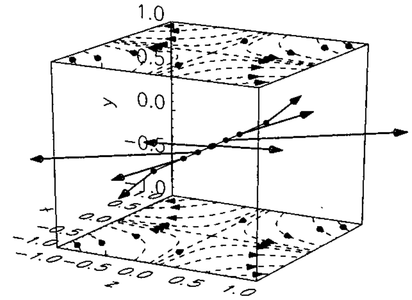
$$p=\beta_e+M_e^2(1-x^2-\alpha_y^2y^2-\alpha_z^2z^2)+1-B_y^2-B_z^2.$$

The pressure is minimum when $y^2=z^2=1$, where the function $f(x)=B_y^2(x)+B_z^2(x)+M_e^2x^2$ is maximum. Denoting this point by x_* then gives the minimum plasma beta as

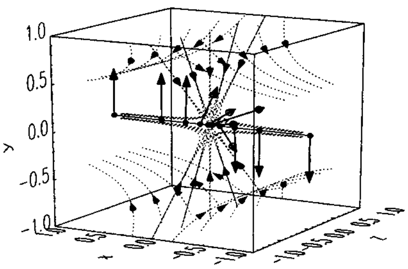
$$\beta_{\min}=M_e^2(x_*^2-2\alpha_y\alpha_z)+B_y^2(x_*)+B_z^2(x_*)-1, \qquad (14)$$

remembering that $\alpha_z=1-\alpha_y$.

In general, x_* and $\beta_{\min}$ depend on M_e and S_e as before, but also on the parameters α_y and α_z (which describe the nature of the flow) and on B_{y0}, B_{z0}, B_{ye} and B_{ze}, the values of the magnetic field components imposed at $x=0$ and $x=L_e$. The case $\alpha_y=1$, $\alpha_z=0$, $B_{y0}=B_{z0}=B_{ze}=0$, $B_{ye}=1$ has already been investigated in Sec. II. Three further specific cases suggested by Sonnerup and Priest (Ref. 2) are now considered. The first consists of another two-dimensional flow, but this time in the x-z plane, with one-dimensional oppositely-directed magnetic field, B_y, perpendicular to the flow being annihilated (Fig. 3). The second case, intermediate between the 2D case of Sec. II and the one just described, has the magnetic field oriented at 45 degrees to the x-z flow at the $x=\pm1$ boundaries (Fig. 4a). The third case consists of an axisymmetric flow, symmetric with respect to rotations about the x-axis, annihilating a two-dimensional field which rotates from being aligned with the z-axis at the origin to pointing in the y-direction at $x=L_e$ (Fig. 4b). This time the magnetic field at the origin is non-



(a)



(b)

FIG. 4. Velocity field (dotted lines) and magnetic field (solid lines anchored on and perpendicular to the line $y=z=0$) for (a) case 2 and (b) case 3c. In both plots $R_m=100$. The magnetic field is everywhere parallel to these latter vectors and the vector length in each plane of constant x is half the dimensionless field strength.

zero in general and so three different illustrative values are considered in this investigation. The parameters used in the different cases are given in Table I.

In the first case, the minimum plasma beta is always M_e^2 because the magnetic field decreases monotonically from B_e at $x=L_e$ to zero at $x=0$ with no local maximum; whenever the maximum occurs at $x=L_e$, Eq. (14) gives the general minimum beta as $M_e^2(1-2\alpha_y\alpha_z)$. Cases 2 and 3, on the other hand, possess local maxima for certain ranges of the Alfvén Mach number and Lundquist number. Cases 2 and 3a have been plotted in Fig. 5. Cases 3b and 3c give beta limits which differ in magnitude but which are visually indistinguishable when plotted on a log-log scale—examination of the graph of case 3c suffices to determine the behavior in these two instances. The low Lundquist number cases have a beta limit of M_e^2 in case 2 and $M_e^2/2$ in cases 3a, 3b and 3c. The values of ξ in all the cases possessing local maxima vary between 1.3 and 2.

TABLE I. Values used for the cases investigated. The magnetic fields have been nondimensionalized with respect to the magnetic field strength at $x=L_e$.

Case	α_y	α_z	B_{y0}	B_{z0}	B_{ye}	B_{ze}
1	0	1	0	0	1	0
2	0	1	0	0	$1/\sqrt{2}$	$1/\sqrt{2}$
3a	1/2	1/2	0	0	1	0
3b	1/2	1/2	0	1/2	1	0
3c	1/2	1/2	0	1	1	0

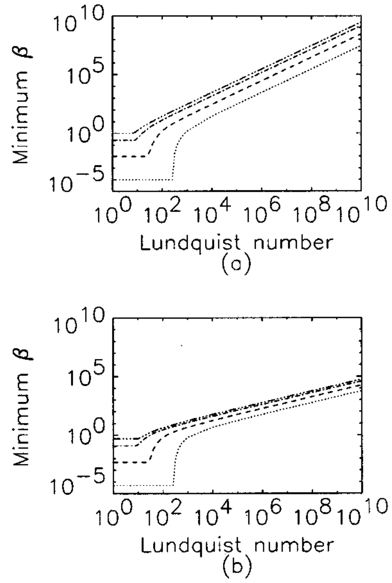


FIG. 5. Minimum plasma beta, $\beta_{\min}$, as a function of the Lundquist number, S_e. The broken curves once again have $M_e=0.01,0.1,0.5$, and 1.0 in ascending order. Part (a) corresponds to case 2 and part (b) to case 3a. Case 1 (not drawn) has a minimum plasma beta of M_e^2, independent of the Lundquist number.

IV. SKEWED TWO-DIMENSIONAL STAGNATION FLOW

Another generalization of the annihilation flow of Sec. II is given in Craig and Henton (Ref. 5). While again restricting the analysis to a two-dimensional situation they have investigated fully two-dimensional velocity and magnetic fields depending on both the x- and y-coordinates and have found a class of solutions satisfying the nonlinear steady-state equations of incompressible resistive MHD. In the notation of this paper their solution is given by the dimensionless variables

$$\mathbf{v}=\alpha x\hat{\mathbf{x}}-\left[\alpha y+\frac{\gamma}{\alpha}ES_e xM\left(1,\frac{3}{2},-\lambda x^2\right)\right]\hat{\mathbf{y}},$$
$$\mathbf{B}=\gamma x\hat{\mathbf{x}}-\left[\gamma y+ES_e xM\left(1,\frac{3}{2},-\lambda x^2\right)\right]\hat{\mathbf{y}}, \tag{15}$$
$$\mathbf{E}=-E\hat{\mathbf{z}},$$

where

$$\lambda=\frac{(\gamma^2-\alpha^2)S_e}{2\alpha}. \tag{16}$$

The variables have been non-dimensionalized with respect to the half-length of the box, L_e, the uniform density, ρ_e, the magnetic field strength at $x=L_e, y=0$ and the Alfvén speed based on this magnetic field strength. Choosing the magnetic field solution to have value B_e at $x=L_e, y=0$ then fixes the magnitude of the uniform electric field strength as

$$E=\frac{(1-\gamma^2)^{1/2}}{S_e M(1,\frac{3}{2},-\lambda)},$$

from which we obtain the restriction

$$-1<\gamma<1 \tag{17}$$

to ensure a real non-zero electric field strength. If $\gamma=\pm 1$ then the velocity and magnetic fields are parallel with no annihilation occurring (this also occurs when $\lambda\to-\infty$). The component of the velocity perpendicular to the magnetic field is

$$\mathbf{v}_\perp=-\frac{(1-\beta^2)^{1/2}(\beta^2-\alpha^2)}{\alpha}[(1-\beta^2)^{1/2}\hat{\mathbf{x}}+\beta\hat{\mathbf{y}}], \tag{18}$$

the magnitude of which is chosen to be v_e at $x=L_e$, corresponding to a dimensionless velocity of magnitude M_e. This choice means that there are two possible values for the parameter α

$$\alpha_\pm=\frac{-(M_e^2+4\gamma^2(1-\gamma^2))^{1/2}\mp M_e}{2(1-\gamma^2)^{1/2}},$$

both of which are negative to ensure an inflow solution. It can be shown that $\lambda>0$ when $\alpha=\alpha_+$ and $\lambda<0$ when $\alpha=\alpha_-$. The behavior of the current density is markedly different in the cases of positive and negative λ. The former corresponds to a current which is maximal in the center and decays towards the edges

$$j_z\simeq\frac{ES_e}{2\lambda x^2}, \tag{19}$$

for x of order 1. The latter solution, on the other hand, corresponds to a situation where the current density grows exponentially as the edges are approached

$$j_z\simeq-ES_e x(-\lambda\pi)^{1/2}\exp(-\lambda x^2). \tag{20}$$

In this situation the electric field strength is no longer approximately $\mathbf{v}\times\mathbf{B}$ and so the Alfvén Mach number is not an appropriate measure of the reconnection rate. Accordingly, the case $\lambda<0$ is not investigated further.

The term reconnective annihilation has been used to describe this model because the solutions combine elements of both annihilation and reconnection solutions. On the one hand, the field lines can be curved when approaching and leaving the X-type neutral point at the origin (the reconnective characteristic) but the solution always possesses an infinitely long current sheet which is aligned with one of the magnetic field separatrices and across which there is no flow (annihilative characteristics). Two examples of these flows are given in Fig. 6.

Craig and Henton non-dimensionalized their pressure as $p'=p\mu_0/B_e^2$ and found that

$$p=p_0-\frac{1}{2}(u_H^2+b_0^2)+\gamma y b_0,$$

where $\mathbf{B}_0=-ES_e xM(1,3/2,-\lambda x^2)\hat{\mathbf{y}}$ and $\mathbf{u}_H=\alpha x\hat{\mathbf{x}}-\alpha y\hat{\mathbf{y}}$. Imposing the pressure at $x=L_e, y=0$ to be p_e then causes p' to be $\beta_e/2$ at this location, leading to the solution

$$p=\frac{1}{2}[\beta_e+1+M_e^2-\gamma^2-g(x,y)], \tag{21}$$

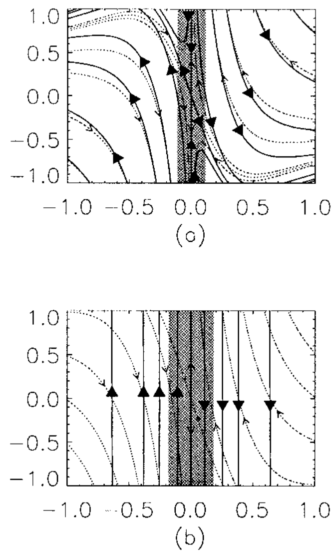


FIG. 6. Stream-lines (dotted) and magnetic field lines (solid) for the skewed 2D flow of Craig and Henton. The shaded region is the area within which the dimensionless current density lies between -10 and -6. The parameters used in part (a) are $\alpha=1$, $\gamma=0.9$ while part (b) has $\alpha=0.1$, $\gamma=0.01$. In both cases the Lundquist number is 100.

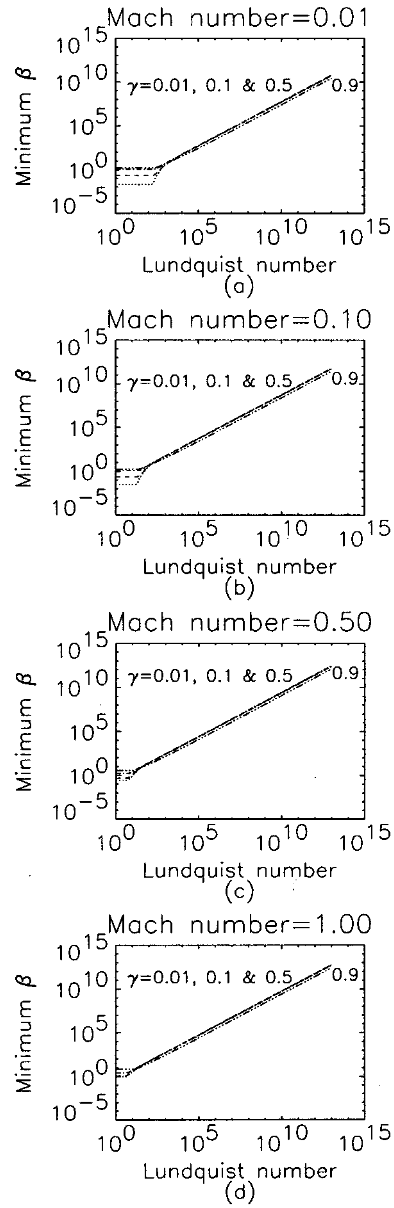


FIG. 7. Variation of the minimum plasma beta with Lundquist number for the fixed Alfvén Mach numbers shown in each graph, but with the skewness parameter, γ, taking the values 0.01 (dotted line), 0.1 (dashed line), 0.5 (dot-dashed line) and 0.9 (three-dots-dashed line).

where

$$g(x,y)=\left[ES_e xM\left(1,\frac{3}{2},-\lambda x^2\right)+\gamma y\right]^2+M_e^2(x^2+y^2)-\gamma^2y^2. \quad (22)$$

In the previous sections it was possible to find the location of the minimum pressure by considering x and y separately since there were no terms involving the product xy, but this is no longer true for the function g. Instead the minimum pressure occurs at the location $x=x_*, y=y_*$ where g is maximum. It can be seen that the situation described here is anti-symmetric with respect to a rotation of π radians about the origin. Since it is the magnitudes of the velocity and magnetic field solutions which determine the value of the minimum pressure the search can be restricted to the region $0\leqslant x\leqslant 1$, $-1\leqslant y\leqslant 1$. For $0<x<1$, $-1<y<1$ a local maximum occurs when $\partial g/\partial x=\partial g/\partial y=0$ and when $\Delta=(\partial^2 g/\partial x^2)(\partial^2 g/\partial y^2)-(\partial^2 g/\partial x\partial y)^2>0$ and either $\partial^2 g/\partial x^2$ or $\partial^2 g/\partial y^2$ is negative. The edges of the region $x=0$ and $x=1$ possess no local maxima, but it is possible for the edges $y=\pm 1$ to possess local maxima. Finally, the function is evaluated at the corner values $(0,-1)$, $(0,1)$, $(1,-1)$, $(1,1)$. Having found the maximum of g, the condition that $p_{\min}>0$ becomes

$$\beta_e>g(x_*,y_*)-1-M_e^2+\gamma^2. \quad (23)$$

The pressure limits for this model are presented in Figs. 7 and 8. There are two competing effects present. The Alfvén Mach number measures the rate of inflow of the plasma at the boundary, which determines the rate at which the oppositely directed magnetic field approaches to annihilate. The larger the Alfvén Mach number, the more likely it is that the magnetic field strength will intensify, forming a boundary layer near the origin. The other factor is the inclination of the inflowing magnetofluid to the magnetic field, measured by the skewness parameter, γ. As $|\gamma|$ increases, the inflow becomes more sheared so that, for fixed M_e, the x-component becomes less significant. The value of the flat segments of each curve in Figs. 7 and 8 can be found by noting that these occur whenever there is no local intensification of the magnetic field within the box considered: the maxima always occur at the corners $(1,1)$ or $(1,-1)$ of the box. Equation (22) then gives

$$g_{\max}=1-\gamma^2+2|\gamma|(1-\gamma^2)^{1/2}+2M_e^2,$$

occurring at $x=1, y=1$ when $\gamma>0$ or $x=1, y=-1$ when $\gamma<0$. Substituting this into Eq. (23) gives

$$\beta_e\geqslant M_e^2+2|\gamma|(1-\gamma^2)^{1/2}. \quad (24)$$

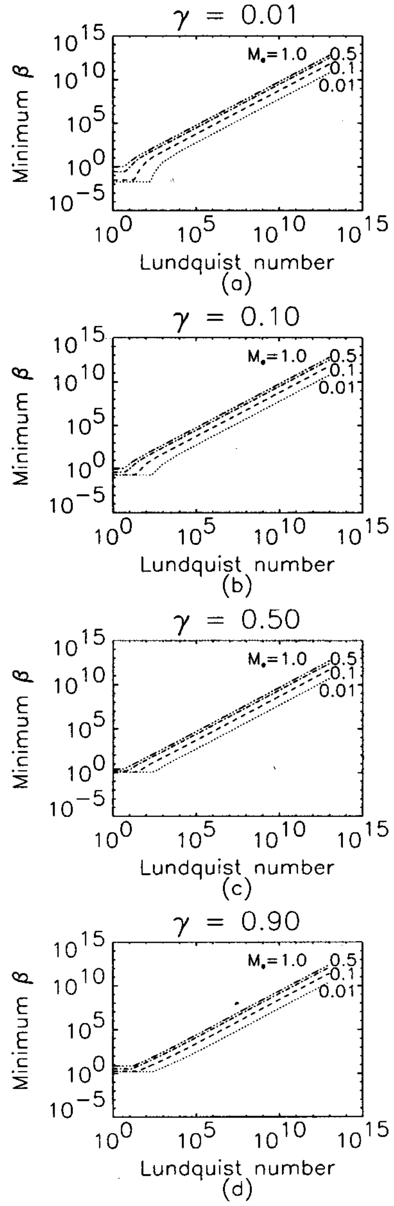


FIG. 8. Variation of the minimum plasma beta with Lundquist number for the fixed skewness parameter, γ, shown in each graph, but with the Alfvén Mach number taking the values 0.01 (dotted line), 0.1 (dashed line), 0.5 (dot-dashed line) and 1.0 (three-dots-dashed line).

It turns out that for all the cases plotted in Figs. 7 and 8 the maximum occurs on one of the lines $y=\pm 1$.

V. CONCLUSIONS

Three problems have been considered in this paper: annihilation of a one-dimensional magnetic field by a two-dimensional stagnation flow, annihilation of a two-dimensional magnetic field by a three-dimensional flow and the reconnective annihilation of a two-dimensional magnetic field by a two-dimensional flow. It is found in all cases that there must be a lower bound for the plasma beta in order to keep the plasma pressure positive everywhere.

In the first annihilation solution the minimum beta for a Lundquist number of 10^6 varies from 6×10^3 to 6×10^6, depending on the Alfvén Mach number. In the second annihilation solution the minimum beta always scales as M_e^2 independently of the Alfvén Mach number for the purely 2D flow with perpendicular field; it lies between 3×10^3 and 3×10^5 for the 2D flow with rotating magnetic field and between 60 and 600 for the axisymmetric flow with a rotating magnetic field. In two dimensions the reconnective annihilation solution has a minimum beta of between 3×10^3 and 6×10^5. In all cases the solutions which permit the lowest minimum values for beta independent of the Lundquist number correspond to weak current sheets of width $2L_e$. However, current sheets of width larger than $R_m^{-1/2}$ do not undergo fast reconnection. Furthermore, it is unlikely that these weak current sheets could be responsible for any appreciable heating of the plasma such as occurs in the solar corona, for example.

The values obtained for the annihilation solutions of Secs. II and III are appropriate for astrophysical situations only when the minimum beta is less than unity. This in turn constrains the solutions to be valid only for Lundquist numbers of less than about 10^3.

The principal cause of the large values for the plasma beta is the use of the incompressible stagnation flow throughout the whole volume of consideration. This linear flow might be expected close to the neutral point but further away it has the undesirable property of continuing to grow in strength as the boundary is approached. In fact, making the region of consideration larger will always increase the minimum beta, the location of which always appears to be on the boundary in the y-direction. The advantage of using this simple linear flow is the availability of analytical solutions to the nonlinear MHD equations, but this paper illustrates the price that must be paid in requiring unrealistically large pressures to compensate for the increase in flow speed toward the boundary.

Fortunately, other viable models do exist for fast reconnection at low plasma beta in astrophysical plasmas. For example, the Almost-Uniform Priest-Forbes (Ref. 6) family of regimes possesses Petschek reconnection as a special case and has weakly curved field lines in the inflow region (see Litvinenko and Forbes (Ref. 7) for a corresponding consideration of pressure limits in this class of solutions). Also, the Non-Uniform Priest-Lee (Ref. 8) family of regimes has strongly curved inflow field lines and has been shown to reproduce well the features of Biskamp's numerical experiment (Ref. 9). Which members of these families occur in practice depends on the geometry, history and boundary conditions. Whereas the annihilation models are completely analytical, these fast reconnection models are much more complex and so naturally rely on a physical matching of their central diffusion regions to the extremal region. Moreover, their existence has been confirmed in numerical experiments when the diffusion region resistivity is enhanced, which is to be expected in many applications due to current-driven micro-instabilities (Refs. 10 and 11). Exploration of the much more difficult three-dimensional process (Refs. 12 and 13) is, however, only in its infancy.

Note added in proof. It has recently been brought to our attention that the requirement that there must be a minimum plasma beta in order that the plasma pressure remain everywhere positive has already been demonstrated for a different set of analytical three-dimensional annihilation solutions possessing vorticity (Ref. 14). Since submitting our paper other work has also appeared in press noting the same result (Ref. 15).

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