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Linear Collapse of Spatially Linear, Two-Dimensional Null Points

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A technique is developed for analysing the linear collapse properties of spatially linear two-dimensional null points with open boundary conditions. A treatment is given of the collapse of nulls which have current and flow so that they are initially in a steady state balance between a magnetic force, a pressure force and a centrifugal force. This extends the previous results for initially current-free X-type nulls with no flow. It is found that all X-points, regardless of the current and flow tend to collapse. Also, O-points collapse in the absence of a plasma flow, but O-points with a large current and possessing a highly super-Alfvénic plasma flow can be stable against linear collapse.

1. Introduction

The surface of the Sun has a temperature of about 6000K. This rises sharply through its atmosphere to the diffuse corona, where the temperature runs into millions of degrees Kelvin. The source of this heating has been a topic of much study with many different theories being proposed. Many such theories invoke *magnetic reconnection* (e.g., Priest and Forbes (2000); Schindler *et al* (1988); Wang and Bhattacharjee (1994)). The Sun is very active magnetically, with regions of flux continually emerging and disappearing through its surface due to plasma motions. These regions of flux give rise to a highly dynamic and complex magnetic field in the corona. Magnetic reconnection occurs when field lines are brought together, break and rejoin with other field lines. This process converts magnetic energy to other forms and could help to explain the coronal heating problem.

Reconnection can often occur where the magnetic field vanishes (a null point). In this paper we study the collapse properties of a linear two-dimensional null point by perturbing it linearly and investigating the exponential time response. Exponential growth indicates the collapse of the null point and an oscillatory response indicates the stability of the null point against collapse from this form of perturbation.

Previous work on the collapse of two-dimensional X-points has been carried out by many people. Dungey (1953) considered a potential X-point with no plasma flow or pressure gradient. He perturbed the magnetic field by moving its footpoints along an external boundary and calculated the effect that this would have on the X-point. He discovered that the X-point would continue its collapse provided the footpoints are still free to move.

Later on, Imshennik and Syrovatsky (1967) started with a linear X-point, added plasma flow and found self-similar solutions for the non-linear evolution of the X-point. They discovered that the magnetic flux function, the plasma density and the plasma velocity would all become singular within a finite time, indicating the collapse of the null point. Further studies of X-point collapse were carried out by Chapman and Kendal (1963) and Forbes and Speiser (1979). Craig and McClymont (1991) also included magnetic diffusion in their analysis.

The incompressible magnetohydrodynamic (MHD) equations (Equations (2.7)-(2.9)) admit non-linear, self-similar solutions, some of which describe collapse. Examples of the two-dimensional cases are described above, and particular three-dimensional solutions have been found by Klapper *et al* (1996) and Bulanov and Sakai (1997). There are also stationary solutions in three dimensions including current and plasma flow (Titov and Hornig (2000)), and so we would like to understand the general behaviour of such a system.

The critical points of a dynamical system are the stationary solutions, and the behaviour of such a system is described by its corresponding phase portrait. The phase portrait near the critical points is described by a linearised system of equations for the dynamical system, and in this paper we use the linearised MHD equations to find the phase portrait for the collapse equations which then provides useful information about some of the possible types of behaviour of the system. The three-dimensional case is formidable, and so we begin here with the two-dimensional case which has not yet been fully studied. We plan to use the resulting techniques to address the more complicated three-dimensional case in future.

In Section 2 we introduce and linearise the MHD equations. We then solve the system of linear equations to find a dispersion relation with which we can create a collapse diagram in Section 5. Section 3 deals with X-points and their properties, from potential X-points to general X-points. Section 4 explores O-points and we discover that they too are liable to collapse under this form of perturbation. Section 5 deals with the effect of plasma flow on null points and in Section 6 we offer our conclusions and ideas for further work.

2. Linear Analysis

2.1. Model Equations

The ideal, time-dependent, incompressible and dimensionless MHD equations are

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = \mathbf{0}, \tag{2.1}$$

and

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{2.2}$$

with the conditions that

$$\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{v} = 0, \tag{2.3}$$

where the magnetic field ($\mathbf{B}$), pressure (p) and plasma velocity ($\mathbf{v}$) are normalised with respect to a characteristic field (B_*), twice the magnetic pressure (B_*^2/μ_0) and the Alfvén speed ($v_A = B_*/(\mu_0 \rho)^{1/2}$), respectively. Time ($t$) is normalised with respect to (l/v_A) in terms of a characteristic length-scale (l).

Equations (2.1)-(2.3) admit a special class of solutions (Klapper *et al*, 1996;

Bulanov & Sakai, 1997) of the form

$$\mathbf{B} = \mathcal{B}(t) \cdot \mathbf{r}, \tag{2.4}$$

$$\mathbf{v} = \mathcal{V}(t) \cdot \mathbf{r}, \tag{2.5}$$

$$p = \frac{1}{2}\mathbf{r}^T \cdot \mathcal{P}(t) \cdot \mathbf{r} + p_0, \tag{2.6}$$

where $\mathbf{r}$ is the position vector and $\mathcal{B}(t), \mathcal{V}(t)$ and $\mathcal{P}(t)$ are time-dependent 2×2 matrices with elements $\mathcal{B}_{ij} = \partial B_i/\partial x_j$, $\mathcal{V}_{ij} = \partial v_i/\partial x_j$ and $\mathcal{P}_{ij} = \partial^2 p/\partial x_i \partial x_j$, respectively. The constant p_0 ensures that the pressure is always positive in the region under consideration.

Solutions of this particular form represent flows of plasma in the neighbourhood of a stagnation point located at a null point ($\mathbf{r} = 0$) of the magnetic field. The solutions will also yield the leading terms in a corresponding Taylor expansion of more general MHD solutions. In particular, one can show that it describes an arbitrary ideal MHD flow in the neighbourhood of a magnetic null (Klapper *et al*, 1996; Bulanov & Sakai, 1997).

Substitution of Equations (2.4)-(2.6) into the MHD equations yields the following matrix system of ordinary differential equations:

$$\frac{d\mathcal{B}}{dt} + \mathcal{B}\mathcal{V} - \mathcal{V}\mathcal{B} = 0, \tag{2.7}$$

$$\frac{d\mathcal{V}}{dt} + \mathcal{V}^2 = -\mathcal{P} + \mathcal{B}^2 - \mathcal{B}^T\mathcal{B}, \tag{2.8}$$

$$tr(\mathcal{B}) = tr(\mathcal{V}) = 0. \tag{2.9}$$

The equations of the magnetic field lines and streamlines are given by $A =$ constant and $\psi =$ constant, respectively, where the flux function (A) and stream function (ψ) are related to the magnetic field and plasma velocity by

$$B_x = \frac{\partial A}{\partial y}, \quad B_y = -\frac{\partial A}{\partial x}, \quad v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}.$$

2.2. Initial State

The initial state ($\mathcal{B}_0$,$\mathcal{V}_0$,$\mathcal{P}_0$) satisfies the steady-state equations

$$\mathcal{B}_0\mathcal{V}_0 - \mathcal{V}_0\mathcal{B}_0 = 0, \tag{2.10}$$

$$\mathcal{V}_0^2 = -\mathcal{P}_0 + \mathcal{B}_0^2 - \mathcal{B}_0^T\mathcal{B}_0, \tag{2.11}$$

with the conditions

$$tr(\mathcal{B}_0) = tr(\mathcal{V}_0) = 0. \tag{2.12}$$

In particular, we shall start with a magnetic field ($\mathcal{B}_0$) of the form

$$\mathcal{B}_0 = \begin{pmatrix} 0 & (1-J_0)/2 \\ (1+J_0)/2 & 0 \end{pmatrix},$$

where J_0 is the dimensionless current in the z-direction. The corresponding flux function is

$$A_0 = \frac{(1-J_0)y^2 - (1+J_0)x^2}{4}.$$

Thus, if $|J_0| < 1$, then $\mathcal{B}_0$ represents an X-point with hyperbolic field lines, whereas $|J_0| > 1$ gives an O-point with elliptical field lines. The special case $|J_0| = 1$ gives a one-dimensional field with straight field lines.

In order to satisfy the induction equation (2.10), we use the only permissible flow ($\mathcal{V}_0$) which is parallel to the field and so has the form

$$\mathcal{V}_0 = \begin{pmatrix} 0 & M_A(1-J_0)/2 \\ M_A(1+J_0)/2 & 0 \end{pmatrix},$$

where $M_A(= v_0/B_0)$ is the dimensionless Alfvén Mach number.

The pressure matrix that is required to balance the magnetic force is then from Equation (2.11)

$$\mathcal{P}_0 = \frac{1}{4}\begin{pmatrix} (J_0+1)(M_A^2(J_0-1)-2J_0) & 0 \\ 0 & (J_0-1)(M_A^2(J_0+1)-2J_0) \end{pmatrix}.$$

2.3. Linearised Equations

In this paper we shall assume that the magnetic field, plasma flow and pressure gradients are made up of the initial, steady component ($\mathcal{B}_0$,$\mathcal{V}_0$,$\mathcal{P}_0$) plus a small, time-dependent perturbation ($\mathcal{B}_1(t)$,$\mathcal{V}_1(t)$,$\mathcal{P}_1(t)$). Linearising Equations (2.7)-(2.9) then reduces them to a matrix system for the perturbed quantities ($\mathcal{B}_1(t)$,$\mathcal{V}_1(t)$,$\mathcal{P}_1(t)$) in terms of the initial state ($\mathcal{B}_0$,$\mathcal{V}_0$,$\mathcal{P}_0$), namely,

$$\frac{d\mathcal{B}_1}{dt} + \mathcal{B}_0\mathcal{V}_1 + \mathcal{B}_1\mathcal{V}_0 - \mathcal{V}_0\mathcal{B}_1 - \mathcal{V}_1\mathcal{B}_0 = 0, \tag{2.13}$$

$$\frac{d\mathcal{V}_1}{dt} + \mathcal{V}_0\mathcal{V}_1 + \mathcal{V}_1\mathcal{V}_0 = -\mathcal{P}_1 + \mathcal{B}_0\mathcal{B}_1 + \mathcal{B}_1\mathcal{B}_0 - \mathcal{B}_0^T\mathcal{B}_1 - \mathcal{B}_1^T\mathcal{B}_0, \tag{2.14}$$

$$tr(\mathcal{B}_1) = tr(\mathcal{V}_1) = 0. \tag{2.15}$$

In order to solve the system of equations, the perturbations ($\mathcal{B}_1$,$\mathcal{V}_1$,$\mathcal{P}_1$) to the initial state are assumed to be of the form

$$\mathcal{B}_1 = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & -B_{11}(t) \end{pmatrix},$$

$$\mathcal{V}_1 = \begin{pmatrix} V_{11}(t) & V_{12}(t) \\ V_{21}(t) & -V_{11}(t) \end{pmatrix},$$

$$\mathcal{P}_1 = \begin{pmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{pmatrix},$$

for which (2.15) is satisfied and $\mathcal{P}_1$ is symmetric. After substituting these matrices into the linearised equations (2.13)-(2.14), we obtain eight equations for the nine unknowns (B_{11}, B_{12}, B_{21}, V_{11}, V_{12}, V_{21}, P_{11}, P_{12}, P_{22}) of the perturbed system. Only seven of these eight equations are independent, however, since the trace of Equation (2.13) automatically vanishes, which means that the two equations in the leading diagonal of the induction equation (2.13) are automatically equal and opposite in sign. This in turn arises because the divergence of Equation (2.1) is satisfied automatically.

The fact that we have only seven independent equations implies that we are free to choose two of the perturbations ourselves, and the rest can then be written in terms of them. Also, it can be shown from these equations that

$$B_{21}(t) = \frac{J_0+1}{J_0-1}B_{12}(t),$$

and

$$V_{21}(t) = V_{12}(t),$$

which further reduces the number of unknown variables to seven and the number of independent equations to five. They are

$$(1+J_0)^2 B_{12} + (J_0-1)\frac{dV_{11}}{dt} + M_A(J_0-1)V_{12} + (J_0-1)P_{11} = 0,$$

$$-J_0 B_{11} + \frac{dV_{12}}{dt} + P_{12} = 0,$$

$$-(J_0-1)B_{12} - \frac{dV_{11}}{dt} + M_A V_{12} + P_{22} = 0,$$

$$\frac{dB_{11}}{dt} + M_A(1+J_0)B_{12} - J_0 V_{12} = 0,$$

$$-M_A(J_0-1)B_{11} + \frac{dB_{12}}{dt} + (J_0-1)V_{11} = 0. \tag{2.16}$$

The components of the field and flow perturbation matrices can be expressed in terms of the pressure components:

$$B_{11}(t) = -\frac{1}{2J_0M_A}\left(\left[1-\frac{6J_0^2-4J_0}{\xi}\right]\frac{dP_{11}}{dt} + \frac{2}{\xi}\frac{d^3P_{11}}{dt^3}\right) + \frac{1}{J_0}\left(P_{12} + \frac{2}{\xi}\frac{d^2P_{12}}{dt^2}\right)$$

$$-\frac{1}{2J_0M_A}\left(\left[1-\frac{6J_0^2+4J_0}{\xi}\right]\frac{dP_{22}}{dt} + \frac{2}{\xi}\frac{d^3P_{22}}{dt^3}\right), \tag{2.17}$$

$$B_{12}(t) = \frac{J_0-1}{2J_0}\left(\frac{-3J_0^2+2J_0}{\xi}P_{11} + \frac{1}{\xi}\frac{d^2P_{11}}{dt^2}\right) - \frac{M_A(J_0-1)}{J_0\xi}\frac{dP_{12}}{dt}$$

$$+\frac{J_0-1}{2J_0}\left(\frac{-3J_0^2-2J_0}{\xi}P_{22} + \frac{1}{\xi}\frac{d^2P_{22}}{dt^2}\right), \tag{2.18}$$

$$V_{11}(t) = -\frac{1}{2J_0}\left(\left[1-\frac{9J_0^2-6J_0}{\xi}\right]\frac{dP_{11}}{dt} + \frac{3}{\xi}\frac{d^3P_{11}}{dt^3}\right) + \frac{M_A}{J_0}\left(P_{12} + \frac{3}{\xi}\frac{d^2P_{12}}{dt^2}\right)$$

$$-\frac{1}{2J_0}\left(\left[1-\frac{9J_0^2+6J_0}{\xi}\right]\frac{dP_{22}}{dt} + \frac{3}{\xi}\frac{d^3P_{22}}{dt^3}\right), \tag{2.19}$$

$$V_{12}(t) = -\frac{1}{2M_A}\left(\left[1-\frac{6J_0^2-4J_0}{\xi}\right]P_{11} + \frac{2}{\xi}\frac{d^2P_{11}}{dt^2}\right) + \frac{2}{\xi}\frac{dP_{12}}{dt}$$

$$-\frac{1}{2M_A}\left(\left[1-\frac{6J_0^2+4J_0}{\xi}\right]P_{22} + \frac{2}{\xi}\frac{d^2P_{22}}{dt^2}\right), \tag{2.20}$$

where $\xi \equiv 4J_0^2 + 3M_A^2J_0^2 - 3M_A^2 - 2$.

After eliminating B_{11}, B_{12}, V_{11} and V_{12} from Equations (2.16), we then obtain a single equation linking the three elements (P_{11}, P_{12}, P_{22}) of the pressure matrix, namely,

$$\frac{d^4P_{11}}{dt^4} + (2J_0 - 2J_0^2 - 1 - M_A^2 + J_0^2M_A^2)\frac{d^2P_{11}}{dt^2} +$$
$$J_0(J_0-1)(J_0^2 - J_0M_A^2 - J_0 - M_A^2)P_{11}$$
$$- \quad 2M_A\left(\frac{d^3P_{12}}{dt^3} - (M_A^2+1)(1-J_0^2)\frac{dP_{12}}{dt}\right) \quad +$$
$$\frac{d^4P_{22}}{dt^4} + (-2J_0 - 2J_0^2 - 1 - M_A^2 + J_0^2M_A^2)\frac{d^2P_{22}}{dt^2} +$$
$$J_0(J_0+1)(J_0^2 - M_A^2 + J_0M_A^2 + J_0)P_{22} = 0. \tag{2.21}$$

Two of the elements of the pressure matrix (say, $P_{12}(t)$ and $P_{22}(t)$) can be considered as being given, so the third ($P_{11}(t)$) is the corresponding solution of this equation. The general solution for $P_{11}(t)$ when $P_{12}(t)$ and $P_{22}(t)$ are given will then be the sum of a particular integral (the driven part) and a complementary function (the normal-mode) part, which contains two arbitrary constants and enables two initial conditions to be satisfied. The complementary function may be found by setting $P_{12}(t) = P_{22}(t) = 0$ in Equation (2.21) and so is the solution of

$$\frac{d^4P_{11}}{dt^4}+(2J_0-2J_0^2-1-M_A^2+J_0^2M_A^2)\frac{d^2P_{11}}{dt^2}+J_0(J_0-1)(J_0^2-J_0M_A^2-J_0-M_A^2)P_{11} = 0,$$

A similar reasoning can be developed if $P_{11}(t)$ and $P_{12}(t)$, say, are imposed instead of $P_{12}(t)$ and $P_{22}(t)$. Because the field and flow perturbations can be expressed in terms of the pressure perturbations, then all of the normal-mode responses will be evident in these perturbation components, so if the pressure perturbations grow, then the rest of the perturbations will do so too, and the null will collapse. Equation (2.21) can be re-written to find the normal-mode response when the pressure perturbations have the form, $P_{11}(t) = P_{11}\exp(\lambda t)$, say, such that a non-collapsing solution will have $\Re(\lambda) = 0$. It becomes

$$\begin{aligned}
(\lambda^4 - (2J_0^2 - 2J_0 + 1 + M_A^2 - J_0^2M_A^2)\lambda^2 + J_0(J_0-1)(J_0^2 - J_0M_A^2 - J_0 - M_A^2))P_{11}& \\
- \quad 2M_A\lambda(\lambda^2 + (M_A^2+1)(J_0^2-1))P_{12} \quad +& \\
(\lambda^4 - (2J_0^2 + 2J_0 + 1 + M_A^2 - J_0^2M_A^2)\lambda^2 + J_0(J_0+1)(J_0^2 + J_0M_A^2 + J_0 - M_A^2))P_{22}& \\
= 0.&
\end{aligned} \tag{2.22}$$

We now proceed to consider the solutions of this equation for a variety of different initial states, namely, X-points, O-points and nulls with flow.

3. X-points

3.1. Current-Free X-point

Starting with the simplest situation, namely, a potential X-point with no initial current or plasma flow ($J_0 = M_A = 0$), Equation (2.22) for the normal-mode behaviour of P_{11} reduces to

$$(\lambda^4 - \lambda^2)P_{11} = 0.$$

The solutions in terms of λ are

$$\lambda = -1, 0, 1.$$

$\lambda = 1$ implies that the perturbations grow exponentially, collapsing the null point. Since these normal mode perturbations can occur whatever the form of the driven pressure (P_{12}, P_{22}), the null point is susceptible to this form of collapse. The normal mode values for P_{12} and P_{22} are just the same.

For the driven problem, the element P_{12} decouples from P_{11} in Equation (2.21). In this case (and the case $J_0 \neq 0$) its time-variation does not affect P_{11}.

With $J_0 = M_A = 0$, Equations (2.16) may be solved to give

$$\begin{aligned} B_{11} &= 0, \\ (1-\lambda^2)B_{12} &= P_{11}, \\ (1-\lambda^2)V_{11} &= \lambda P_{11}, \\ \lambda V_{12} &= -P_{12}, \\ P_{22} &= -P_{11}. \end{aligned} \tag{3.1}$$

The pressure perturbation then takes the form

$$p_1 = \frac{P_{11}}{2}(x^2 - y^2) + P_{12}xy,$$

and so on a circle of radius 1, say, where $x = \cos\theta$ and $y = \sin\theta$, it becomes

$$p_1 = \frac{P_{11}\cos(2\theta) + P_{12}\sin(2\theta)}{2}.$$

We can therefore impose odd or even boundary conditions (or a combination of them) on the pressure perturbation along the circle $r = 1$, say. Imposing just the odd condition, by putting $P_{11} = 0$ and $P_{12} \neq 0$, we find from Equations (3.1) that $\lambda^2 = 1$ if $B_{12} \neq 0$, which means that the perturbations can grow exponentially and the X-point collapses with a growth rate $\lambda = 1$. Thus, during the collapse, $B_{11} = P_{22} = P_{11} = 0$, $B_{12} = -B_{21} = V_{11}$, $V_{12} = V_{21} = -P_{12}$ and the direction of the collapse depends on the value of V_{11}: if $V_{11} < 0$ then $B_{12} < 0$ and the inclination of the separatrix $y = x$ in the first quadrant to the x-axis increases in value and vice versa. If we instead impose the even condition, by putting $P_{12} = 0$ and $P_{11} \neq 0$ with $V_{12} \neq 0$, then we find a state of marginal stability ($\lambda = 0$).

Alternatively, we could instead impose boundary conditions on the unit circle, say, on the magnetic field rather than the pressure. The unperturbed flux function is

$$A_0 = (y^2 - x^2)/2 \quad (= \cos(2\theta_0)/2 \quad \text{on the boundary}), \tag{3.2}$$

while the perturbed flux function is

$$A_0 + A_1 = (y^2 - x^2)/2 + B_{12}(x^2 + y^2), \tag{3.3}$$

or, on the boundary ($r = 1$),

$$A_0 + A_1 = \cos(2\theta_1)/2 + B_{12}. \tag{3.4}$$

Now suppose we move a field line (with flux function $A_0 = \cos(2\theta_0)/2$) from an angular position θ_0 on the unit circle to a position $\theta_1 = \theta_0 + \Delta\theta$ (with flux function $A = \cos(\theta_1)/2 + B_{12}$). In an ideal motion the value of A is conserved and so after linearising we find

$$\Delta\theta = \frac{B_{12}}{\sin(2\theta_0)}. \tag{3.5}$$

In the first quadrant, $\sin(2\theta_0) > 0$ and so, if $B_{12} > 0$, $\Delta\theta$ is positive and the X-point closes up towards the y-axis, whereas if $B_{12} < 0$ it collapses towards the x-axis.

3.2. The Symmetric Case

If we constrain $B_{11} = 0$, $V_{12} = 0$, $P_{12} = 0$ and $M_A = 0$, then the perturbations will not rotate the null and will collapse it to either the x or the y- axis. Equations (2.16) then give the relations

$$V_{11} = \frac{1}{4J_0}\left(\frac{dP_{11}}{dt} + \frac{dP_{22}}{dt}\right), \tag{3.6}$$

$$B_{12} = \frac{1-J_0}{4J_0}\left(P_{11} + P_{22}\right), \tag{3.7}$$

$$\left(\frac{d^2P_{11}}{dt^2} - (J_0-1)^2 P_{11}\right) + \left(\frac{d^2P_{22}}{dt^2} - (J_0+1)^2 P_{22}\right) = 0. \tag{3.8}$$

This will give pressure growth rates of $\pm(J_0+1)$ or $\pm(J_0-1)$. This will collapse the null point, as there is always a positive, real value for the growth rate.

3.3. X-points with Current but No Flow

3.3.1. Physical Cause of Collapse of an X-point with Current. Next, consider a magnetic field with current of the form

$$B_{0x} = \frac{y}{\alpha_0}, \quad B_{0y} = \alpha_0 x\,, \tag{3.9}$$

whose field lines are described by

$$\frac{y^2}{\alpha_0} - \alpha_0 x^2 = k, \tag{3.10}$$

and k is a constant. The resulting current is along the z-direction, is uniform and has a value

$$j_0 = \frac{\alpha_0^2 - 1}{\mu_0 \alpha_0}.$$

The Lorentz force due to this field is

$$\mathbf{j_0} \times \mathbf{B_0} = j_0 \begin{pmatrix} -\alpha_0\, x \\ y/\alpha_0 \end{pmatrix},$$

which balances the pressure gradient in the equilibrium state. The Ohmic heating due to this current is

$$\frac{(\alpha_0^2-1)^2}{\sigma \mu_0^2 \alpha_0^2}. \tag{3.11}$$

Now suppose we perturb the field so its components become

$$B_x = \frac{y}{\alpha}, \quad B_y = \alpha x\,,$$

in place of (3.9) and the same field lines are described by

$$\frac{y^2}{\alpha} - \alpha x^2 = k,$$

in place of (3.10). The resulting current is

$$j = \frac{\alpha^2 - 1}{\mu_0 \alpha},$$

Figure 1. A plot of the magnetic field lines of an X-point with the initial state $\alpha_0 = 1$ (dotted curves) and the perturbed state having $\alpha = 1.2$ (solid curves). The arrows show the direction and relative magnitude of the additional force acting on the plasma.

again in the z-direction, so that the Lorentz force due to the new field is

$$\mathbf{j} \times \mathbf{B} = j \left(\begin{array}{c} -\alpha\, x \\ y/\alpha \end{array} \right).$$

The Ohmic heating is now

$$\frac{(\alpha^2 - 1)^2}{\sigma \mu_0^2 \alpha^2}. \tag{3.12}$$

The additional Lorentz force acting on the plasma is

$$\mathbf{F} = \frac{\alpha^2 - \alpha_0^2}{\mu_0} \left(\begin{array}{c} -x \\ y/\alpha_0^2 \alpha^2 \end{array} \right).$$

Figure 1 shows an initial X-point (dotted), a perturbed X-point and the additional magnetic force that is acting on the plasma in the perturbed state. This force is in such a direction as to continue the collapse of the X-point. For example, along the x-axis it is directed inwards, towards the null point due to the enhanced magnetic pressure force and along the y-axis it is directed outwards, away from the null point due to the enhanced magnetic tension force. Of course, there are other forces (such as a pressure gradient) that could in principal act against the magnetic force.

From (3.11) and (3.12), we can also see that if α is further away in value from 1 than α_0, so that the X-point collapses, then the heating will increase. If, on the other hand, α is closer in value to 1 than α_0, so that the angle between the separatrices approaches $\pi/2$, the heating will decrease.

3.3.2. Linear Analysis. Keeping $M_A = 0$, but now allowing J_0 to be non-zero so there can be some initial current in the null point, Equation (2.22) becomes

$$\begin{aligned} &(\lambda^4 - (2J_0^2 - 2J_0 + 1)\lambda^2 + J_0^2(J_0 + 1)^2)P_{11} + \\ &\qquad\qquad (\lambda^4 - (2J_0^2 + 2J_0 + 1)\lambda^2 + J_0^2(J_0 - 1)^2)P_{22} = 0. \end{aligned}$$

If we set $P_{22} = 0$ and solve for the normal modes (λ) of P_{11}, then we find that the

Figure 2. The growth rates (λ) of the pressure perturbations P_{11} (*left*) and P_{22} (*right*) as functions of the dimensionless current (J_0).

growth rates for P_{11} are $\lambda = \pm J_0$, $\pm(J_0 - 1)$. Instead, setting $P_{11} = 0$ allows us to calculate the growth rates for the normal modes of P_{22}, which are $\lambda = \pm J_0$, $\pm(J_0 + 1)$. All of these represent an exponential growth of the perturbation, and therefore a collapse of the null point but with different growth rates from before. These are shown in Figure 2 which implies that for $-1 < J_0 < 1$ there is always a positive, non-zero value of λ, giving exponential growth of the perturbation, regardless of the value of J_0. The effect of increasing the current from zero is to decrease the growth-rate of the stable mode and increase that of the neutral mode.

This analysis applies to all types of null points with no flow, namely, X-points ($|J_0| < 1$), O-points ($|J_0| > 1$) and one-dimensional current sheets ($|J_0| = 1$).

4. O-points

4.1. Physical Cause of Collapse

The linear analysis of the previous section suggests that an O-point will tend to collapse (when there is no plasma flow present), which, as far as we are aware, is a new result, so what is the physical cause for this collapse?

Consider a magnetic field ($\mathbf{B}$) such that

$$B_{0x} = -\frac{1}{\alpha_0}\, y, \quad B_{0y} = \alpha_0\, x,$$

whose field lines are elliptical and are described by

$$\frac{1}{\alpha_0}\, y^2 + \alpha_0\, x^2 = k,$$

where k is a constant and the area of the ellipse is πk and so is independent of α_0. The current (j_0) is given by

$$j_0 = \frac{\alpha_0^2 + 1}{\mu_0 \alpha_0},$$

in the z-direction. The Lorentz force is

$$\mathbf{j_0} \times \mathbf{B_0} = j_0 \begin{pmatrix} -\alpha_0\, x \\ -y/\alpha_0 \end{pmatrix},$$

which can be balanced by the pressure gradient so that the null point is in equilibrium. The Ohmic heating (j^2/σ) is given by

$$\frac{(\alpha_0^2 + 1)^2}{\sigma \mu_0^2 \alpha_0^2}$$

Now perturb the field such that

$$B_x = -\frac{1}{\alpha}\, y, \quad B_y = \alpha\, x,$$

and the same field lines are described by

$$\frac{1}{\alpha}\, y^2 + \alpha\, x^2 = k,$$

and have a constant area as α changes. The current (j) is given by

$$j = \frac{\alpha^2 + 1}{\mu_0 \alpha},$$

and the Lorentz force is

$$\mathbf{j} \times \mathbf{B} = j \begin{pmatrix} -\alpha\, x \\ -y/\alpha \end{pmatrix}.$$

The additional magnetic force acting on the plasma is therefore

$$\mathbf{F} = \frac{\alpha^2 - \alpha_0^2}{\mu_0} \begin{pmatrix} -x \\ y/(\alpha^2 \alpha_0^2) \end{pmatrix},$$

which will tend to carry on flattening the ellipse as can be seen in Figure 3. The magnetic tension force ($[\mathbf{B} \cdot \nabla]\mathbf{B}/\mu_0$) is stabilizing and the magnetic pressure force ($\nabla(B^2)/[2\mu_0]$) is destabilizing and dominant, causing the collapse due to magnetic forces. The Ohmic heating (j^2/σ) is now given by

$$\frac{(\alpha^2 + 1)^2}{\sigma \mu_0^2 \alpha^2},$$

from which it can be seen that, if α is further away from 1 in value than α_0, the ellipse is flattened and the heating rises. If, on the other hand, α is closer in value to 1 than α_0, the ellipse becomes more circular and the Ohmic heating reduces. This simple analysis shows that a magnetic null in the form of an O-point tends to collapse into a current sheet under the present approximations due to the magnetic forces. Other forces (again, such as a pressure gradient) could, in principal, stop the collapse.

Figure 3. A plot of the magnetic field lines of an O-point with $\alpha_0 = 1$ (dotted ellipses) and the perturbed field with $\alpha = 1.2$ (solid ellipses). The arrows indicate the direction and relative strength of the additional force acting on the plasma.

5. Effect of flow

Equation (2.22) can be used to find the normal-mode frequencies of the system by setting two of the elements to zero and requiring that the coefficient of the third be zero. This produces three equations, the solutions of which give us the normal-mode frequencies for P_{11}, P_{12} and P_{22}, respectively,

$$\lambda^4+(-2J_0^2+2J_0-1-M_A^2+J_0^2M_A^2)\lambda^2+J_0(J_0-1)(J_0^2-J_0M_A^2-J_0-M_A^2) = 0, \quad (5.1)$$

$$\lambda^2 + (M_A^2 + 1)(J_0^2 - 1) = 0, \quad (5.2)$$

$$\lambda^4+(-2J_0^2-2J_0-1-M_A^2+J_0^2M_A^2)\lambda^2+J_0(J_0+1)(J_0^2+J_0M_A^2+J_0-M_A^2) = 0. \quad (5.3)$$

Figure 4 shows the curves in M_A-J_0 parameter space where $\Re(\lambda)$ vanishes for each of the normal modes. The figure also indicates where P_{11}, P_{12} and P_{22} are each unstable (i.e., at least one value of $\Re(\lambda)$ is positive) or stable (i.e., all values of $\Re(\lambda)$ are zero) to collapse. It is constructed in the following way. The polynomials (5.1) and (5.3) both have the form

$$\lambda^4 + A\lambda^2 + B = 0,$$

where A and B are real, and so

$$\lambda^2 = \frac{-A \pm (A^2 - 4B)^{1/2}}{2}$$

If $B < 0$, the discriminant in this solution is positive, giving one positive and one negative solution for λ^2, which means that at least one value of $\Re(\lambda)$ is positive and the corresponding pressure parameter grows exponentially in time. If $A < 0$, the solution for λ^2 will have a positive real part, which means that again at least one of the values of $\Re(\lambda)$ will be positive and the pressure grows. If $A^2 - 4B < 0$, the solution for λ^2 is complex, which implies that at least one of the values of $\Re(\lambda)$ is positive, and again we have a collapsing solution. The condition for perturbations that will not cause the collapse of the null point is that $\Re(\lambda) = 0$, so there is a pure oscillation (i.e., $\lambda^2 < 0$) which therefore occurs when all of A, B and $A^2 - 4B$

Figure 4. Plots of the regions in J_0-M_A parameter space where different perturbations have growing solutions. The sets of three letters indicate the stability to collapse of the pressure elements in that particular region. Thus, for instance, "uss" would indicate that P_{11} in unstable to collapse, whereas P_{12} and P_{22} are both stable to collapse. The top figure shows one completely stable region. The bottom figure shows the region for $0 < J_0 < 1$ and $0 < M_A < 1$, i.e., an X-point.

are positive. In Figure 4, therefore, we have drawn the curves $A = 0$, $B = 0$ and $A^2 - 4B = 0$ for Equations (5.1) and (5.3), which provide the corresponding crossover locations between regions of collapsing and non-collapsing solutions for P_{11} and P_{22}. Furthermore, Equation (5.2) for P_{12} has the form

$$\lambda^2 + C = 0,$$

where C is real. In Figure 4 we have therefore also plotted the curve $C = 0$ which gives the crossover points between collapsing and non-collapsing solutions for P_{12}.

There is only one completely stable region, where all of $A, B, C, A^2 - 4B$ are positive and all the values of λ are purely imaginary so that P_{11}, P_{12} and P_{22} are all purely oscillatory. It is located in the upper right part of the top diagram in Figure 4. The other regions have at least one of the pressure perturbations growing exponentially and so collapsing the null point provided that the boundary conditions are free. Since the stable region is in the area where $J_0 > 7$ and $M_A > 2$, we can see that all X-points ($|J_0| < 1$) and O-points with weak current ($1 < J_0 < 7$) are unstable to collapse. For most physically realistic flows (i.e., $M_A < 1$ with sub-Alfvénic plasma velocities) the null point will tend to collapse as a result of these perturbations. It is only O-points with strongly super-Alfvénic flows that can be stable against collapse.

Choosing a value from the stability region, namely $[J_0 = 15, M_A = 3]$, substituting these values into the equations for the normal mode solution of P_{11} (5.1) and setting $P_{11} = P_{22} = 0$, we can show that the extra forces due to the pressure $(-\mathcal{P}_1)$, the magnetic pressure $(-\mathcal{B}_0^T\mathcal{B}_1 - \mathcal{B}_1^T\mathcal{B}_0)$, the magnetic tension $(\mathcal{B}_0\mathcal{B}_1 + \mathcal{B}_1\mathcal{B}_0)$ and the plasma velocity $(-\mathcal{V}_0\mathcal{V}_1 - \mathcal{V}_1\mathcal{V}_0)$ are stabilizing when taken together.

Using the above values for J_0 and M_A, we can obtain as a particular example the following expressions for the pressure perturbations.

$$P_{11} = 2\sin(39.8\,t) + 2\sin(2.96\,t),$$

$$P_{12} = P_{22} = 0.$$

Focussing on the x-component of the forces at the point (1,0), and looking at a small time $t = 10^{-8}$, such that we catch the very initial effect of the perturbations, we find that the pressure force is $-8.556768091 \times 10^{-7}$, the magnetic pressure force is $+2.230974646 \times 10^{-6}$, the magnetic tension force is negligible at $+2 \times 10^{-16}$ and the centrifugal force is $+1.66396063 \times 10^{-7}$. The magnetic pressure dominates, and the total is $+1.5416939 \times 10^{-6}$. This is acting outwards, away from the centre of the elliptical field line. The x-component of the velocity perturbation at this point is -0.2378326164 - in towards the centre of the field line. The force is acting against the perturbed velocity which indicates that it is acting to stabilize the plasma flow.

The only force acting to destabilize the plasma flow is the pressure force. The other forces at this point are all acting so as to support the flow of plasma and keep the O-point from collapse.

6. Conclusions

In this paper, we have developed an elegant method to consider the linear collapse of a linear null point to spatially linear perturbations. We have discovered that all X-type null points including steady flows and current have a tendancy to collapse. An O-point without flow will also collapse, but an O-point with strong enough flow can be stable.

If the spatially linear system extends to infinity then its energy is infinite. If it is truncated by a boundary at some radius then the boundary is not in general closed, so that a collapse is associated with an inflow of energy into the region. However, if the linear system under study here is regarded as the local behaviour near a null point in a more global configuration, then the solutions indicate a tendency for collapse and therefore magnetic dissipation and heating near the null in response to distant motions.

In future papers we shall aim to develop the ideas further by considering the nonlinear behaviour near null points and the much more complex structure of MHD collapse near three-dimensional null points.

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References

Bulanov, S. V. and Sakai, J. 1997 Magnetic collapse in incompressible plasma flows. *J. Phys. Soc. Jap.* **66**, 3477–3483.

Chapman, S. and Kendall, P. C. 1963 Liquid instability and energy transformation near a magnetic neutral line: A soluble non-linear hydromagnetic problem. *Proc. Roy. Soc. Lond.* **271**, 435–448.

Craig, I. J. D. and McClymont, A. N. 1991 Dynamic magnetic reconnection at an X-type neutral point. *Astrophys. J.* **371**, L41–L44.

Dungey, J. W. 1953 Conditions for the occurrence of electrical discharges in astrophysical systems. *Phil. Mag.* **44**, 725–738.

Forbes, T. G. and Speiser, T. W. 1979 Temporal evolution of magnetic reconnection in the vicinity of a magnetic neutral line. *J. Plasma Phys.* **21**, 107–126.

Imshennik, V. S. and Syrovatsky, S. I. 1967 Two-dimensional flow of an ideally conducting gas in the vicinity of the zero line of a magnetic field. *Sov. Phys. JETP* **25**, 656–664.

Klapper, I., Rado, A. and Tabor, M. 1996 A Lagrangian study of dynamics and singularity formation at magnetic null points in ideal three-dimensional magnetohydrodynamics. *Phys. Plasmas* **3**, 4281–4283.

Priest, E. R. and Forbes, T. G. 2000 *Magnetic Reconnection.* Cambridge University Press.

Schindler, K., Hesse, M. and Birn, J. 1988 General magnetic reconection, parallel electric fields and helicity. *J. Geophys. Res.* **93**, 5547–5557.

Titov, V. S. and Hornig, G. 2000 Magnetohydrodynamic flows sustaining stationary magnetic nulls. *Phys. Plasmas* **7**, 3542–3550.

Wang, X. and Bhattacharjee, A. 1994 Current sheets and reconnection driven by footpoint motion in two-dimensional coronal loops with X-type neutral lines. *Ap. J.* **420**, 415–421.