School of Mathematics and Statistics.

MT 4510 - Solar Theory - Sheet 10.

CEP Sem 2, 2005/06.

1. Jan 2000 Q3

   (a) N/A

   (b) Solve the equations of continuity and momentum for a steady, spherically symmetric, isothermal solar wind to show that its speed (v) at a distance r from the solar centre is determined by

$$\frac{v^2}{c_{si}^2} - \log\left(\frac{v^2}{c_{si}^2}\right) = 4\log\left(\frac{r}{r_c}\right) + 4\frac{r_c}{r} + C$$

where c_{si}, r_c and C are constants. [5]

First right down equations:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\rho\mathbf{v}) &= 0, \qquad \text{- mass continuity}\\
\rho\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v} &= -\boldsymbol{\nabla}p + \rho\mathbf{g}, \qquad \text{- momentum}\\
p &= \frac{\rho\mathcal{R}T}{\tilde{\mu}}, \qquad \text{- gas law}\\
T &= T_0. \qquad \text{- energy}
\end{aligned}$$

Velocity is purely radial: $\mathbf{v} = v\hat{\mathbf{r}}$ and gravity obeys the inverse square law, $\mathbf{g} = -(GM_\circ/r^2)\hat{\mathbf{r}}$.

In spherical coordinates, and with a steady flow, mass continuity becomes

$$\frac{d}{dr}\left(r^2\rho v\right) = 0 \qquad \Rightarrow \qquad \rho = \frac{D}{r^2 v}.$$

where D is a constant.

Radial component of momentum:

$$\rho v\frac{dv}{dr} = -\frac{dp}{dr} - \frac{GM_\circ\rho}{r^2}.$$

Define isothermal sound speed as $c_{si} = (p/\rho)^{1/2}$, so gas law gives

$$p = c_{si}^2\rho.$$

By substituting in the gas law, and then mass continuity, the momentum can be written in terms of v and r only

$$\begin{aligned}
v\frac{dv}{dr} &= -\frac{c_{si}^2}{\rho}\frac{d\rho}{dr} - \frac{GM_\circ}{r^2}.\\
v\frac{dv}{dr} &= -c_{si}^2 r^2 v\frac{d}{dr}\left(\frac{1}{r^2 v}\right) - \frac{GM_\circ}{r^2}.\\
&= \frac{c_{si}^2}{v}\frac{dv}{dr} + \frac{2c_{si}^2}{r} - \frac{GM_\circ}{r^2}.
\end{aligned}$$

Rearranging

$$\left(v - \frac{c_{si}^2}{v}\right)\frac{dv}{dr} = 2\frac{c_{si}^2}{r^2}\left(r - r_c\right),$$

where $r_c = GM_\circ/2c_{si}^2$.
Integrating the above gives

$$\begin{aligned}
\int v - \frac{c_{si}^2}{v}dv &= 2c_{si}^2\int \frac{1}{r} - \frac{r_c}{r^2}dr, \\
\frac{c_{si}^2}{2}\left(\left(\frac{v}{c_{si}}\right)^2 - \log v^2\right) &= 2c_{si}^2\left(\log r + \frac{r_c}{r}\right) + C' \\
\left(\frac{v}{c_{si}}\right)^2 - \log\left(\frac{v}{c_{si}}\right)^2 &= 4\log\left(\frac{r}{r_c}\right) + 4\frac{r_c}{r} + C,
\end{aligned}$$

where C and C' are constants, as required.

(c) Sketch the solutions of this equation and deduce the behaviour of v and the plasma pressure at large distances for both the 'solar wind' and 'solar breeze' solutions. Why was the solar wind solution thought to be more relevant? Suggest extra effects that could be included to make the model more realistic. [8]

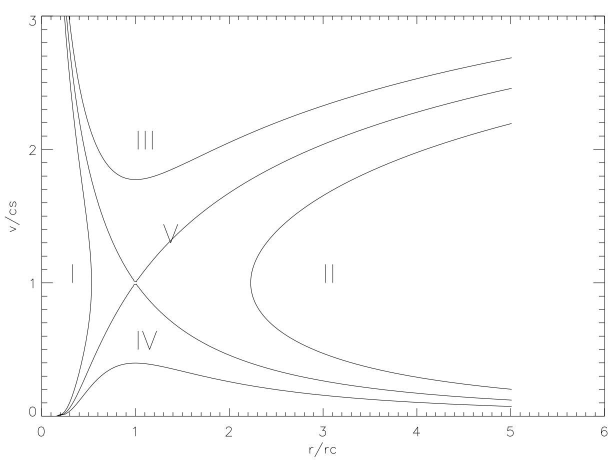

Figure 1: The solar wind velocity, v, as a function of the radius, r for various values of the constant C. The solar wind solution is solution V and the solar breeze solution is IV.

- Solution IV - solar breeze solution: gives $v \rightarrow 0$ as $r \longrightarrow \infty$, so as $r \longrightarrow \infty$ solution approximated by

$$-\log\left(\frac{v}{c_{si}}\right)^2 \approx 4\log\left(\frac{r}{r_c}\right) \qquad \Rightarrow \qquad \frac{v}{c_{si}} \approx \left(\frac{r_c}{r}\right)^2 \qquad \Rightarrow \qquad r^2 v \approx r_c^2 c_{si}$$

Thus, mass continuity gives

$$\rho = \frac{D}{r^2 v} = \frac{D}{r_c^2 c_{si}} = \text{ const.}$$

Since the density tends to a constant value and the plasma is isothermal, $p = c_{si}^2 \rho$, so will the pressure. Thus, the solar breeze solution is unphysical since it cannot be contained by the extremely small interstellar pressure.

- Solution V - solar wind solution: gives large v as $r \longrightarrow \infty$, so as $r \longrightarrow \infty$ solution approximated by

$$\left(\frac{v}{c_{si}}\right)^2 \approx 4 \log\left(\frac{r}{r_c}\right) \qquad \Rightarrow \qquad \frac{v}{c_{si}} \approx 2\sqrt{\log\left(\frac{r}{r_c}\right)}.$$

Thus, density is

$$\rho = \frac{D}{r^2 v} \approx \frac{D'}{r^2\sqrt{\log(r/r_c)}}.$$

So

$$\rho \to 0 \qquad \text{as} \qquad r \to \infty.$$

Since the plasma is isothermal, so $p = c_{si}^2 \rho$, and the pressure also tends to zero. This means that the solution can eventually match onto the interstellar plasma at large distances from the Sun, so is a more physically realistic model of the solar wind.

Extra effects that may be considered are

- Drop the isothermal constraint and use an alternative energy equation, e.g.

$$p = K\rho^\gamma.$$

- Drop the spherically symmetric constraint and consider super-radial expansion of the wind.