HONOURS M. A. AND HONOURS B. Sc. EXAMINATION

MATHEMATICS AND STATISTICS

Paper MT4510 : Solar Theory

May 2005

Time allowed : Two hours

Attempt ALL questions

The following MHD Equations, written using the usual notation, may be quoted and used where required

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \qquad \rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g},$$

$$p = 2\rho \mathrm{R} T, \qquad \mathbf{j} = \nabla \times \mathbf{B}/\mu, \qquad \nabla \cdot \mathbf{B} = 0,$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \qquad \frac{1}{\sigma}\mathbf{j} = \mathbf{E} + \mathbf{v} \times \mathbf{B}.$$

The following may be useful:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}),$$

$$\nabla(\mathbf{A}^2) = 2(\mathbf{A} \cdot \nabla)\mathbf{A} + 2\mathbf{A} \times (\nabla \times \mathbf{A}),$$

$$(1 - x^2)^{-1} = 1 - x^2 + x^4 - x^6 + ..., \qquad x^2 < 1.$$

1. (a) Derive the induction equation stating clearly the assumptions involved and the equations used. Define the magnetic Reynolds number (R_m) and describe briefly the behaviour of the magnetic field when (i) $R_m \gg 1$ and (ii) $R_m \ll 1$. [5]

The following equations are used:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \qquad \mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{\sigma}\mathbf{j}, \qquad \mathbf{j} = \nabla \times \mathbf{B}/\mu, \qquad \nabla \cdot \mathbf{B} = 0.$$

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E}, \\ &= -\nabla \times \left(\frac{1}{\sigma}\mathbf{j} - \mathbf{v} \times \mathbf{B}\right), \\ &= -\nabla \times \left(\frac{1}{\sigma\mu}(\nabla \times \mathbf{B})\right) + \nabla \times (\mathbf{v} \times \mathbf{B}),\end{aligned}$$

Assume $\eta = 1/(\sigma\mu)$ *is constant. So*

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \eta\nabla(\nabla \cdot \mathbf{B}) + \eta\nabla^2\mathbf{B},$$

but $\nabla \cdot \mathbf{B} = 0$, *hence*

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta\nabla^2\mathbf{B},$$

Magnetic Reynolds number, R_m,

$$R_m = \frac{\nabla \times (\mathbf{v} \times \mathbf{B})}{\eta\nabla^2\mathbf{B}} = \frac{v_0 B_0 L_0^2}{\eta L_0 B_0} = \frac{v_0 L_0}{\eta},$$

If $R_m \gg 1$, *then magnetic field is 'frozen-in' to the plasma, i.e., plasma and field move together.*
If $R_m \ll 1$, *then magnetic field diffuses through plasma.*

(b) Consider a horizontal magnetic field of the form $\mathbf{B} = (B(y,t), 0, 0)$ and an imposed velocity of the form

$$\mathbf{v} = (x\frac{\mathrm{d}v(y)}{\mathrm{d}y}, -v(y), 0).$$

Show that the induction equation reduces to

$$\frac{\partial B}{\partial t} = -\frac{\partial}{\partial y}(vB) + \eta\frac{\partial^2 B}{\partial y^2}.$$

[2]

$$\begin{aligned}
\mathbf{v} \times \mathbf{B} &= (0, 0, v(y)B) \\
\nabla \times (\mathbf{v} \times \mathbf{B}) &= (\frac{\partial(v(y)B}{\partial y}, 0, 0) \\
\nabla^2 \mathbf{B} &= (\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2})(B(y,t), 0, 0) = (\frac{\partial B(y,t)}{\partial y^2}, 0, 0) \\
\frac{\partial \mathbf{B}}{\partial t} &= (\frac{\partial B(y,t)}{\partial t}, 0, 0)
\end{aligned}$$

$\Rightarrow$ *induction equation becomes*

$$\frac{\partial B(y,t)}{\partial t} = \frac{\partial(v(y)B)}{\partial y} + \eta \frac{\partial B(y,t)}{\partial y^2}.$$

(c) Write down the ideal form of the induction equation obtained in (b). The, using the method of separation of variables, find $B(y,t)$ subject to the initial condition

$$B(y,0) = \frac{B_0}{1-y^2},$$

where B_0 is a constant, given that $v(y) = y$. Verify that the solution is

$$B(y,t) = \frac{B_0 e^t}{1+(ye^t)^2}.$$

[6]

Ideal induction equation:

$$\frac{\partial B(y,t)}{\partial t} = \frac{\partial(v(y)B)}{\partial y}, \qquad \text{and} \qquad v(y) = y.$$

So

$$\frac{\partial B(y,t)}{\partial t} = y\frac{\partial B}{\partial y} + B.$$

Using separation of variables, write $B(y,t) = Y(y)T(t)$ *so*

$$Y\frac{\mathrm{d}T}{\mathrm{d}t} = y\frac{\mathrm{d}Y}{\mathrm{d}y}T + YT$$

$$\Rightarrow \qquad \frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{y}{Y}\frac{\mathrm{d}Y}{\mathrm{d}y} + 1 = \alpha = constant$$

therefore,

$$\int \frac{\mathrm{d}T}{T} = \int \alpha \mathrm{d}t, \qquad \Rightarrow \qquad T = Ae^{\alpha t},$$

and

$$\int \frac{\mathrm{d}Y}{Y} = \int (\alpha - 1)y\mathrm{d}y, \qquad \Rightarrow \qquad Y = Cy^{\alpha-1}.$$

So

$$B(y,t) = \sum_{\alpha} A_\alpha e^{\alpha t} y^{\alpha-1}.$$

Applying initial condition and expanding:

$$B(y,0) = \sum_{\alpha} A_\alpha y^{\alpha-1} = \frac{B_0}{1-y^2} = B_0(1 - y^2 + y^4 - y^6 + ...).$$

$$\begin{aligned}
\alpha &= 1: \quad A_1 y^0 = B_0 \quad \Rightarrow \quad A_1 = B_0, \\
\alpha &= 3: \quad A_3 y^2 = -B_0 y^2 \quad \Rightarrow \quad A_3 = -B_0, \\
\alpha &= 5: \quad A_5 y^4 = B_0 y^4 \quad \Rightarrow \quad A_5 = B_0, \\
\alpha &= 7: \quad A_7 y^6 = -B_0 y^6 \quad \Rightarrow \quad A_7 = -B_0,
\end{aligned}$$

Therefore, $\alpha = 2n+1, n = 0, 1, 2, ...$
So

$$B(y,0) = B_0 \sum_{n=0} (-1)^n y^{(2n+1)-1} = B_0 \sum_{n=0} (-1)^n y^{2n},$$

and

$$B(y,t) = B_0 \sum_{n=0} (-1)^n e^{(2n+1)t} y^{2n} = B_0 e^t \sum_{n=0} (-1)^n (e^t y)^{2n}.$$

Therefore

$$B(y,t) = \frac{B_0 e^t}{1 + (ye^t)^2}.$$

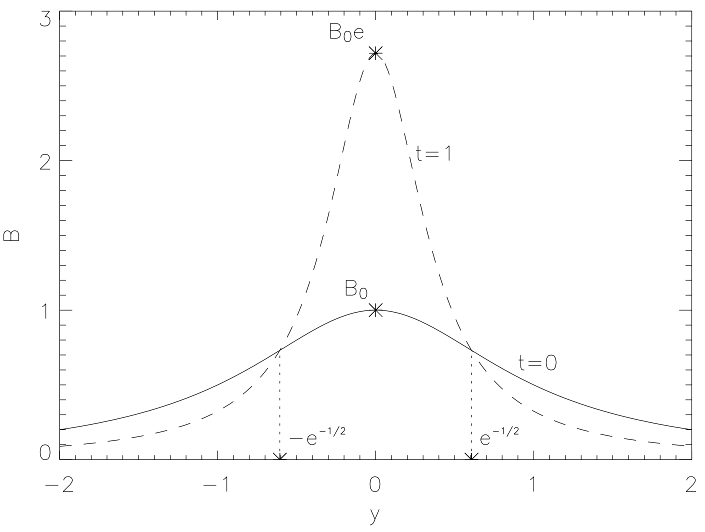
(d) Sketch the curves of B versus y at times $t = 0$ and $t = 1$. [2]
At $t = 0$, $B(y,0) = B_0/(1+y^2)$,
So at $y = 0$, $B(0,0) = B_0$ *and as* $y = \pm\infty$, $B(\pm\infty, 0) \to 0$.

At $t = 1$, $B(y,1) = B_0 e/(1+(ey)^2)$,
So at $y = 0$, $B(0,0) = B_0 e$ *and as* $y = \pm\infty$, $B(\pm\infty, 1) \to 0$.

The curves cross when $B(y,1) = B(y,0)$,

$$\Rightarrow \qquad \frac{B_0 e}{1+(ey)^2} = \frac{B_0}{1+y^2} \qquad \Rightarrow \qquad (e^2 - e)y^2 = e(e-1)y^2 = e - 1,$$

So when $y = \pm e^{-1/2}$.



2. (a) Given that a plasma of uniform temperature (T_0) is in equilibrium under a balance between a pressure gradient and gravity in a vertical magnetic field, find its pressure $p(z)$ as a function of height (z) in terms of the pressure p_0 at the base ($z = 0$); identify the scale height Λ. [5]

Gas law: $p(z) = 2\rho(z)\mathcal{R}T_0$.
Equation of motion: $\nabla p(z) = \rho(z)\mathbf{g}$.

$$\mathbf{g} = -g\hat{\mathbf{z}}, \qquad p(0) = p_0$$

So

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g = -\frac{pg}{2\mathcal{R}T_0}.$$

$$\int \frac{1}{p}\mathrm{d}p = -\int \frac{g}{2\mathcal{R}T_0}\mathrm{d}z$$

$$\Rightarrow \quad \log p = -\frac{g}{2\mathcal{R}T_0}z + \log C,$$

$$\Rightarrow p = Ce^{-gz/(2\mathcal{R}T_0)},$$

when $z = 0, \;\; p(0) = C = p_0$.

$$\text{So } p = p_0 e^{-gz/(2\mathcal{R}T_0)} = p_0 e^{-z/\Lambda}, \text{ where } \Lambda = 2\mathcal{R}T_0/g.$$

(b) Assuming that pressure scale height is large ($\Lambda \gg L$), where L is the typical length scale in the system, reduce the equation of motion to

$$\mathbf{j} \times \mathbf{B} = 0,$$

stating clearly the conditions under which this can occur in terms of the Alfvén speed v_A, the plasma beta, β and a typical velocity, $v_0 = L/\tau$, where τ is a typical timescale in the system. [4]

[See over

Equation of motion assuming $\Lambda \gg L$, i.e., with $\mathbf{g}$ neglected :

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \frac{1}{\mu}(\nabla\times\mathbf{B})\times\mathbf{B}$$

$$\underset{(1)}{\rho_0\frac{v_0}{\tau}} + \underset{(2)}{\rho_0\frac{v_0^2}{L}} = \underset{(3)}{-\frac{p}{L}} + \underset{(4)}{\frac{1}{\mu}\frac{B^2}{L}}.$$

Compare (3) and (4):

$$\frac{p}{L} \ll \frac{1}{\mu}\frac{B^2}{L} \qquad \text{if} \qquad \frac{\mu p}{B^2} \ll 1 \qquad \Rightarrow \qquad \beta/2 \ll 1.$$

So can neglect pressure gradient in favour of Lorentz force if $\beta \ll 1$.
Compare (1) or (2) and (4):

$$\rho_0\frac{v_0}{\tau} \ll \frac{1}{\mu}\frac{B^2}{L} \qquad \text{if} \qquad v_0^2 \ll \frac{B^2}{\mu\rho_0} \qquad \Rightarrow \qquad v_0 \ll v_A.$$

So can neglect derivatives of $\mathbf{v}$ in favour of Lorentz force if $v_0 \ll v_A$.

(c) Show that the Lorentz force can be written as

$$\mathbf{j}\times\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla(B^2/2).$$

[2]

From $\mathbf{j} = (\nabla\times\mathbf{B})/\mu$, and identity given on front page,

$$\begin{aligned}\mathbf{j}\times\mathbf{B} &= \frac{1}{\mu}(\nabla\times\mathbf{B})\times\mathbf{B}\\ &= -\frac{1}{\mu}(\frac{1}{2}\nabla(B^2) - (\mathbf{B}\cdot\nabla)\mathbf{B})\\ &= (\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla(B^2/2)\end{aligned}$$

(d) Consider the magnetic field $\mathbf{B} = (x-by, bx-y, 0)$. For what value of the constant b are the magnetic pressure and tension forces in balance? [2]
Magnetic pressure and tension forces balance if $\mathbf{j}\times\mathbf{B} = \mathbf{0}$.

$$\mathbf{B} = (x-by, bx-y, 0), \qquad \text{and} \qquad \mathbf{j} = \frac{1}{\mu}(0,0,2b).$$

$$\mathbf{j}\times\mathbf{B} = \frac{2b}{\mu}(-bx+y, x-by, 0) = \mathbf{0}\ \ \forall x, y, \ \text{when}$$

$$2b(-bx+y) = 0 \qquad \text{and} \qquad 2b(x-by) = 0 \qquad \Rightarrow \qquad b = 0.$$

So $\mathbf{j}\times\mathbf{B} = \mathbf{0}$ when $b = 0$.

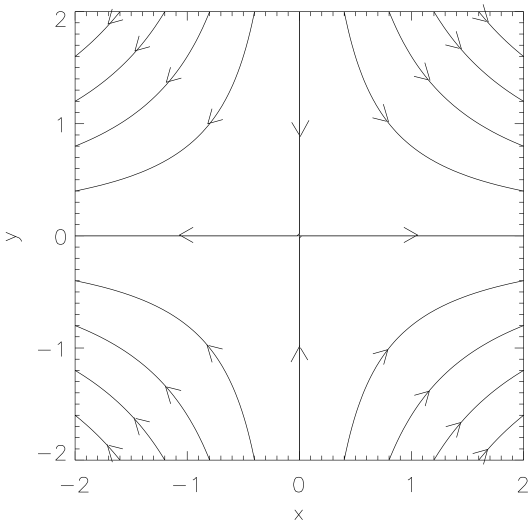
(e) Sketch the field lines for the magnetic field in part (d) if (i) $b = 0$ and (ii) $b = 1$. [6]

(i) $b = 0, \quad \Rightarrow \quad \mathbf{B} = (x, -y, 0)$.

Fieldlines:

$$\int \frac{\mathrm{d}x}{x} = -\int \frac{\mathrm{d}y}{y} \quad \Rightarrow \quad \log x = -\log y + \log C \quad \Rightarrow \quad xy = C.$$

If $C = 0$, *then* $x = 0$ *and* $y = 0$ *are field lines.*
If $C \neq 0$, *then as* $x \to \infty$, $y \to C/x \to 0$.
If $C \neq 0$, *then as* $x \to 0$, $y \to C/x \to \infty$.
Spacing: $B = \sqrt{x^2 + y^2}$, *therefore as* x *or* y *get large the field lines are closer together.*
Arrows: $\mathbf{B} = (x, -y, 0)$.
If $x = 0$ *then* $B_y = -y$ *so arrows on* y*-axis are directed in towards the origin. If* $y = 0$ *then* $B_x = x$ *so arrows on* x*-axis are directed away from the origin.*



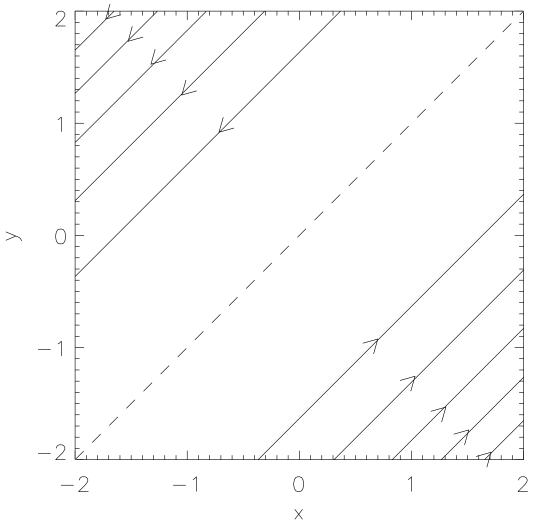
(ii) $b = 1, \quad \Rightarrow \quad \mathbf{B} = (x - y, x - y, 0)$.

Fieldlines:

$$\int \mathrm{d}x = \int \mathrm{d}y \quad \Rightarrow \quad x = y + C \quad \Rightarrow \quad x - y = C.$$

If $C = 0$, *then* $x = y$.
$C \neq 0$ *gives lines parallel to* $x = y$.
Spacing: $B = \sqrt{2}|x - y|$. *Therefore when* $x = 0$, $B = \sqrt{2}y$ *and so as* y *increases so does* B *and the lines get closer together. Similarly, when* $y = 0$, $B = \sqrt{2}x$ *and so as* x *increases so does* B.
Arrows: $\mathbf{B} = (x - y, x - y, 0)$.
When $x = 0$ *then* $B_x = B_y = -y$ *so arrows when* $y > 0$ *are directed downwards and when* $y < 0$ *are directed upwards.*

[See over



3. (a) Consider a uniform plasma with

$$p = p_0, \qquad \rho = \rho_0 \quad \text{and} \quad \mathbf{B} = (0, 0, B_0),$$

where p_0, ρ_0 and B_0 are all constants. Given that there are no pressure or density variations and gravity is absent, linearised about the uniform state and, hence, obtain the linear, ideal, MHD equations. [4]
Variables:

$$\begin{aligned} p &= p_0, \quad p_1 = 0 \\ \rho &= \rho_0, \quad \rho_1 = 0 \\ \mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_1(\mathbf{r}, t), \\ \mathbf{v} &= \mathbf{v}_1(\mathbf{r}, t). \end{aligned}$$

Linearised ideal, MHD Equations in absence of gravity:

$$\begin{aligned} \rho_0 \nabla \cdot \mathbf{v}_1 &= 0, \qquad \Rightarrow \qquad \nabla \cdot \mathbf{v}_1 = 0, \\ \rho_0 \frac{\partial \mathbf{v}_1}{\partial t} &= \frac{1}{\mu} (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0, \\ \frac{\partial \mathbf{B}_1}{\partial t} &= \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0), \\ \nabla \cdot \mathbf{B}_1 &= 0. \end{aligned}$$

(b) Assuming that the perturbations in velocity and magnetic field are proportional to $\exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t)$ use the equations obtained in (a) to derive the dispersion relation for Alfvén waves. [6]
Differentiate equation of motion with respect to t:

$$\rho_0 \frac{\partial^2 \mathbf{v}_1}{\partial t^2} = \frac{1}{\mu} (\nabla \times \frac{\partial \mathbf{B}_1}{\partial t}) \times \mathbf{B}_0.$$

Use ideal induction equation to replace $\partial \mathbf{B}_1/\partial t$,

$$\rho_0 \frac{\partial^2 \mathbf{v}_1}{\partial t^2} = \frac{1}{\mu}(\nabla \times (\nabla \times (\mathbf{v}_1 \times \mathbf{B}_0))) \times \mathbf{B}_0.$$

$$\mathbf{v}_1 = \mathbf{C}_v \exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t) \qquad \text{and} \qquad \mathbf{B}_1 = \mathbf{C}_B \exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t).$$

So

$$\frac{\partial}{\partial t} = -i\omega, \qquad \text{and} \qquad \frac{\partial^2}{\partial t^2} = -\omega^2.$$

$$\nabla\cdot = i\mathbf{k}\cdot, \qquad \text{and} \qquad \nabla\times = i\mathbf{k}\times .$$

Fourier decomposing equation:

$$-\rho_0\omega^2\mathbf{v}_1 = \frac{1}{\mu}(i\mathbf{k} \times (i\mathbf{k} \times (\mathbf{v}_1 \times \mathbf{B}_0))) \times \mathbf{B}_0.$$

$$\rho_0\omega^2\mathbf{v}_1 = \frac{1}{\mu}(\mathbf{k} \times (\mathbf{k} \times (\mathbf{v}_1 \times \mathbf{B}_0))) \times \mathbf{B}_0.$$

Building up right-hand side:

$$\mathbf{k} \times (\mathbf{v}_1 \times \mathbf{B}_0) = (\mathbf{k}\cdot\mathbf{B}_0)\mathbf{v}_1 - (\mathbf{k}\cdot\mathbf{v}_1)\mathbf{B}_0 = (\mathbf{k}\cdot\mathbf{B}_0)\mathbf{v}_1,$$

since $\nabla\cdot\mathbf{v}_1 = \mathbf{k}\cdot\mathbf{v}_1 = 0$.

$$\mathbf{k} \times (\mathbf{k} \times (\mathbf{v}_1 \times \mathbf{B}_0)) = (\mathbf{k}\cdot\mathbf{B}_0)(\mathbf{k}\times\mathbf{v}_1).$$

$$\begin{aligned}
[\mathbf{k} \times (\mathbf{k} \times (\mathbf{v}_1 \times \mathbf{B}_0))] \times \mathbf{B}_0 &= (\mathbf{k}\cdot\mathbf{B}_0)[(\mathbf{k}\times\mathbf{v}_1)\times\mathbf{B}_0] \\
&= (\mathbf{k}\cdot\mathbf{B}_0)[(\mathbf{B}_0\cdot\mathbf{k})\mathbf{v}_1 - (\mathbf{B}_0\cdot\mathbf{v}_1)\mathbf{k}] \\
&= (\mathbf{k}\cdot\mathbf{B}_0)^2\mathbf{v}_1,
\end{aligned}$$

since $\mathbf{v}_1\cdot\mathbf{B}_0 = 0$ *from equation of motion.*
Hence,

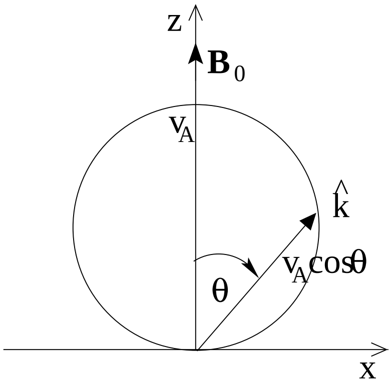
$$\rho_0\omega^2\mathbf{v}_1 = (\mathbf{k}\cdot\mathbf{B}_0)^2\mathbf{v}_1/\mu, \quad \Rightarrow \quad \omega^2 = \frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{\mu\rho_0}. \tag{1}$$

Define, $v_A = B_0^2/\sqrt{\mu\rho_0}$, *and use* $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ *and* $\mathbf{k} = (k_x, k_y, k_z)$, *so dispersion relation for Alfven waves is*

$$\omega = \pm v_A(\mathbf{k}\cdot\hat{\mathbf{z}}) = \pm v_A k_z.$$

(c) What are the main characteristics of Alfvén waves? Sketch their phase speed on a polar diagram. [5]

Main characteristics of Alfven waves:
– alfven waves are anisotropic because of $\mathbf{k}\cdot\mathbf{B}_0$ *term.*
– $\mathbf{v}_1 \perp \mathbf{B}_0$ *and* $\mathbf{v}_1 \perp \mathbf{k}$ *therefore they are transverse waves*
– no disturbances in pressure or density so $\nabla\cdot\mathbf{v}_1 = 0$, *so waves incompressible.*



$\mathbf{k}\cdot\hat{\mathbf{z}} = k\cos\theta$, *so phase speed,* $c_{ph} = \omega/k = v_A\cos\theta$.
when $\theta = 0$, $c_{ph} = v_A$,
when $\theta = \pi/4$, $c_{ph} = v_A/\sqrt{2}$,
when $\theta = \pi/2$, $c_{ph} = 0$.
Polar diagram: circle centre $(\theta = 0, v_A/2)$ *and radius* $v_A/2$.