HONOURS M. A. AND HONOURS B. Sc. EXAMINATION

MATHEMATICS AND STATISTICS

Paper MT4510 : Solar Theory

May 2005

Time allowed : Two hours

Attempt ALL questions

The following MHD Equations, written using the usual notation, may be quoted and used where required

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \qquad \rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g},$$

$$p = 2\rho \mathrm{R} T, \qquad \mathbf{j} = \nabla \times \mathbf{B}/\mu, \qquad \nabla \cdot \mathbf{B} = 0,$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \qquad \frac{\mathbf{j}}{\sigma} = \mathbf{E} + \mathbf{v} \times \mathbf{B}.$$

The following may be useful:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C},$$

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}),$$

$$\nabla(\mathbf{A} \cdot \mathbf{A}) = 2(\mathbf{A} \cdot \nabla)\mathbf{A} + 2\mathbf{A} \times (\nabla \times \mathbf{A}),$$

$$(1 + x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \ldots , \qquad x^2 < 1.$$

[See over

1. (a) Derive the induction equation stating clearly the assumptions involved and the equations used. Define the magnetic Reynolds number (R_m) and describe briefly the behaviour of the magnetic field when (i) $R_m \gg 1$ and (ii) $R_m \ll 1$. [6]

(b) Consider a horizontal magnetic field of the form $\mathbf{B} = (B(y,t), 0, 0)$ and an imposed velocity of the form

$$\mathbf{v} = \left(x\frac{\mathrm{d}v(y)}{\mathrm{d}y}, -v(y), 0 \right).$$

Show that the induction equation reduces to

$$\frac{\partial B}{\partial t} = \frac{\partial}{\partial y}(vB) + \eta \frac{\partial^2 B}{\partial y^2}. \qquad [2]$$

(c) Write down the ideal form of the induction equation obtained in (b) supposing that $v(y) = y$. Then, using the method of separation of variables or otherwise, find $B(y,t)$ subject to the initial condition

$$B(y,0) = \frac{B_0}{1+y^2},$$

where B_0 is a constant. Verify that the solution is

$$B(y,t) = \frac{B_0 e^t}{1+(ye^t)^2}. \qquad [6]$$

(d) Sketch the curves of B versus y at times $t = 0$ and $t = 1$. [2]

2. (a) Given that a plasma of uniform temperature (T_0) is in equilibrium under a balance between a pressure gradient and gravity in a vertical magnetic field, find its pressure $p(z)$ as a function of height (z) in terms of the pressure p_0 at the base ($z = 0$); identify the pressure scale height Λ. [5]

(b) Assuming that the scale height is large ($\Lambda \gg L$), where L is the typical length scale in the system, reduce the equation of motion to

$$\mathbf{j} \times \mathbf{B} = \mathbf{0},$$

stating clearly the conditions under which this can occur in terms of the Alfvén speed v_A, the plasma beta β, and a typical velocity $v_0 = L/\tau$, where τ is a typical timescale in the system. [4]

(c) Show that the Lorentz force can be written as

$$\mathbf{j} \times \mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{B}/\mu - \nabla(B^2/2\mu). \qquad [2]$$

(d) Consider the magnetic field $\mathbf{B} = (x - by, bx - y, 0)$. For what value of the constant b are the magnetic pressure and tension forces in balance? [2]

(e) Sketch the field lines for the magnetic field in part (d) if (i) $b = 0$ and (ii) $b = 1$. [6]

3. (a) Consider a uniform plasma with

$$p = p_0, \qquad \rho = \rho_0 \quad \text{and} \quad \mathbf{B} = (0, 0, B_0),$$

where p_0, ρ_0 and B_0 are all constants. Given that there are no pressure or density variations and gravity is absent, linearise about the uniform state and, hence, obtain the linear, ideal, MHD equations. [4]

(b) Assuming that the perturbations in velocity and magnetic field are proportional to $\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$, use the equations obtained in (a) to derive the dispersion relation for Alfvén waves. [6]

(c) What are the main characteristics of Alfvén waves? Sketch their phase speed on a polar diagram. [5]