Chapter 0

Handout

0.1 Notation

Throughout this course we will be using scalars, vectors and matrices. It is essential that you know what they are and can tell the difference between them!!

- *Scalar:* e.g. α, β or γ. Scalars belong to a field F such as $\mathbb{R}$ or $\mathbb{C}$.
- *Vectors:* e.g. $\mathbf{x}$ or $f(x)$. The first vector belongs to a Vector Space (defined later) such as $\mathbb{R}^n$, $\mathbb{C}^n$ and is of the form $\mathbf{x} = (x_i, x_2, \ldots, x_n)$. The second vector is a continuous function such as the polynomial $x^2 + x$ and belongs to a vector space such as $C(-\infty, \infty)$.

  In the lectures, vectors will be denoted by:

  $$\underset{\sim}{x} - \text{i.e. a small letter with a wiggly line under}$$

  and in the online lecture notes by

  $$\mathbf{x} - \text{i.e. a bold small letter .}$$

- *Matrices:* e.g. $\mathbf{A}$ or $\mathbf{B}$. An $n \times n$ matrix has the form

  $$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \ldots & \ldots & a_{nn} \end{bmatrix}.$$

  In the lectures, a matrix will be denoted by:

  $$\underline{A} - \text{a capital letter with a straight line under}$$

and in the online lecture notes by

$$\mathbf{A} - \text{i.e. a bold capital letter} .$$

0.2 Introduction

This Numerical Analysis course will cover the derivation, description and analysis of obtaining numerical solutions to mathematical problems. To be able to select the best (most appropriate) method for a particular problem we need an understanding of the underlying theoretical principles behind each method. So on this course we will do very little actual computer programming.

The primary theme of this course is approximation. The purpose of approximation is to make a difficult problem easier, or more mathematically, tractable. We will consider some of the standard methods, algorithms and ideas associated with approximation. Obviously, when approximating, the answer is unlikely to be the same as the exact answer and it is important to know the error associated with the approximate answer. We will deal with errors arising due to *rounding* (i.e. due to a limit in the number of digit spaces for a number stored digitally) and the *termination of an infinite process.* These errors are unavoidable and are not mistakes. We would like these errors to be small. An obvious way is to work at higher accuracy, but this requires more computation and so there is a trade off. In this course, we will look at how to calculate the error bounds for each method. To do this we need to look at ways of measuring sensitivity, which is the first part of this course.

For example, a typical approach to many "continuous" problems, such as finding a solution of an ordinary differential equation, is to create a "discrete" model of the equations and generate a Linear System. In theory, linear equations are easily solved. You will have probably all met *Gaussian elimination*, which is a method for producing a solution to a linear system in a fixed, and pre-determined, number of steps. However, Gaussian elimination does not necessarily produce the correct answer. For example, we start by wanting to find $\mathbf{x}$ where

$$\mathbf{A}\mathbf{x} = \mathbf{b} .$$

The process by which the equation is solved actually generates $\tilde{\mathbf{x}}$, which is the correct answer, $\mathbf{x}$, plus the accumulated effect of all the rounding errors that have taken place during the solution process, $\delta\mathbf{x}$,

$$\tilde{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x} .$$

A key question is whether the discrepancy between what we want, $\mathbf{x}$, and what we get, $\tilde{\mathbf{x}}$, is small? Furthermore, how sensitive is the solution (i.e. the error) to small changes in the input? That is is the problem well- or ill-conditioned?

This course will cover four topics:

- Norms (way to measure errors) – lecturer: Dr Régnier
- Iterative methods to solve linear systems of equations – lecturer: Dr Parnell
- Approximations to functions – lecturer: Dr Régnier
- Best approximations – lecturer: Dr Parnell

First we will revise a few basic terms:

Vector Space, Basis, Norm, and Inner Product.

0.3 Books and Website

It is not essential to by a book for this course, but if you want to look at one the following covers most of the material in this course:

"Theory & Applications of Numerical Analysis"

by G.M. Phillips & P.J. Taylor

Academic Press, 2nd Edition, 1996

All the lecture notes will be put up on the web. They will go up a chapter at a time, *after* each chapter has been covered in the lectures.

All tutorials and tutorial solutions will also go online. The tutorial sheets will go up the week they are handed out and the solutions will go up the week *after* the tutorial for that sheet.

The lecture notes and tutorial sheets can be found here:

http://www-solar.mcs.st-and.ac.uk/~clare/Lectures/num-analysis.html