Chapter 4

Best Approximation

4.1 The General Case

In the previous chapter, we have seen how an interpolating polynomial can be used as an approximation to a given function. We now want to find the best approximation to a given function.

This fundamental problem in Approximation Theory can be stated in very general terms. Let V be a Normed Linear Space and W a finite-dimensional subspace of V, then for a given $\mathbf{v} \in V$, find $\mathbf{w}^* \in W$ such that

$$\|\mathbf{v} - \mathbf{w}^*\| \leq \|\mathbf{v} - \mathbf{w}\| ,$$

for all $\mathbf{w} \in W$. Here $\mathbf{w}^*$ is called the *Best Approximation* to $\mathbf{v}$ out of the subspace W. Note that the definition of V defines the particular norm to be used and, when using that norm, $\mathbf{w}^*$ is the vector that is closest to $\mathbf{v}$ out of all possible vectors in W. In general, different norms lead to different approximations.

In the context of Numerical Analysis, V is usually the set of continuous functions on some interval $[a, b]$, with some selected norm, and W is usually the space of polynomials P_n. The requirement that W is finite-dimensional ensures that we have a basis for W.

Least Squares Problem

Let $f(x)$ be a given particular continuous function. Using the *2-norm*

$$\|f(x)\|_2 = \left(\int_a^b f^2(x)dx \right)^{1/2}$$

find $p^*(x)$ such that

$$\|f(x) - p^*(x)\|_2 \leq \|f(x) - p(x)\|_2 ,$$

for all $p(x) \in P_n$, polynomials of degree at most n, and $x \in [a, b]$.

This is known as the **Least Squares Problem**. Best approximations with respect to the 2-norm are called **least squares approximations**.

4.2 Least Squares Approximation

In the above problem, how do we find $p^*(x)$? The procedure is the same, regardless of the subspace used.

So let W be any finite-dimensional subspace of dimension $(n+1)$, with basis vectors

$$\phi_0(x),\ \phi_1(x), \dots \text{ and } \phi_n(x)\,.$$

Therefore, any member of W can be expressed as

$$\Psi(x) = \sum_{i=0}^{n} c_i \phi_i(x)\ ,$$

where $c_i \in \mathbb{R}$. The problem is to find c_i such that $\|f - \Psi\|_2$ is **minimised**.

Define

$$E(c_0, c_1, \dots, c_n) = \int_a^b (f(x) - \Psi(x))^2 dx\,.$$

We require the minimum of $E(c_0, c_1, \dots, c_n)$ over all values $c_0, c_1, \dots, c_n$. A necessary condition for E to have a minimum is:

$$\begin{aligned}\frac{\partial E}{\partial c_i} = 0 &= -2\int_a^b (f - \Psi)\frac{\partial \Psi}{\partial c_i} dx\,,\\ &= -2\int_a^b (f - \Psi)\phi_i(x) dx\,.\end{aligned}$$

This implies,

$$\int_a^b f(x)\phi_i(x)dx = \int_a^b \Psi\phi_i(x)dx\,,$$

or

$$\int_a^b f(x)\phi_i(x)dx = \int_a^b \sum_{j=0}^{n} c_j\phi_j(x)\phi_i(x)dx\,.$$

Hence, the c_i that minimise $||f(x) - \Psi(x)||_2$ satisfy the system of equations given by

$$\int_a^b f(x)\phi_i(x)dx = \sum_{j=0}^{n} c_j \int_a^b \phi_j(x)\phi_i(x)dx, \qquad \text{for } i = 0, 1, \dots, n\,, \tag{4.1}$$

a total of $(n+1)$ equations in $(n+1)$ unknowns $c_0, c_1, \dots, c_n$.

These equations are often called the Normal Equations.

Example 4.2.1 *Using the Normal Equations (4.1) find the $p(x) \in P_n$ the best fits, in a least squares sense, a general continuous function $f(x)$ in the interval $[0, 1]$.*

i.e. find $p^*(x)$ such that

$$\|f(x) - p^*(x)\|_2 \le \|f(x) - p(x)\|_2 \,,$$

for all $p(x) \in P_n$, polynomials of degree at most n, and $x \in [0, 1]$.

Take the basis for P_n as

$$\phi_0 = 1, \phi_1 = x, \phi_2 = x^2, \ldots, \phi_n = x^n \,.$$

Then

$$\begin{aligned}
\int_0^1 f(x) x^i dx &= \sum_{j=0}^{n} c_j \int_0^1 x^j x^i dx \\
&= \sum_{j=0}^{n} c_j \int_0^1 x^{i+j} dx \\
&= \sum_{j=0}^{n} c_j \left[\frac{x^{i+j+1}}{i+j+1} \right]_0^1 \\
&= \sum_{j=0}^{n} \frac{c_j}{i+j+1} \,.
\end{aligned}$$

Or, writing them out:

$$\begin{aligned}
i = 0 : &\qquad \int_0^1 f dx = c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \cdots + \frac{c_n}{n+1} \\
i = 1 : &\qquad \int_0^1 x f dx = \frac{c_0}{2} + \frac{c_1}{3} + \frac{c_2}{4} + \cdots + \frac{c_n}{n+2} \\
\ldots & \\
i = n : &\qquad \int_0^1 x^n f dx = \frac{c_0}{n+1} + \frac{c_1}{n+2} + \cdots + \frac{c_n}{2n+1} \,.
\end{aligned}$$

Or, in matrix form:

$$\begin{bmatrix} 1 & 1/2 & \ldots & 1/n+1 \\ 1/2 & 1/3 & \ldots & 1/n+2 \\ \vdots & \vdots & \ddots & \vdots \\ 1/n+1 & 1/n+2 & \ldots & 1/2n+1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \int_0^1 f(x) dx \\ \int_0^1 x f(x) dx \\ \vdots \\ \int_0^1 x^n f(x) dx \end{bmatrix}$$

Does anything look familiar? A system $\mathbf{HA} = \mathbf{f}$ where $\mathbf{H}$ is the Hilbert matrix. This is seriously bad news - this system is famously ILL-CONDITIONED! We will have to find a better way to find p^*.

4.3 Orthogonal Functions

In general, it will be hard to solve the Normal Equations, as the Hilbert matrix is ill-conditioned. The previous example is an example of what not to do!

Instead, using the same approach as before choose (if possible) an orthogonal basis $\phi_i(x)$ such that

$$\int_a^b \phi_i(x)\phi_j(x)dx = 0, \qquad i \neq j\,.$$

In this case, the Normal Equations (4.1) reduce to

$$\int_a^b f(x)\phi_i(x)dx = c_i \int_a^b \phi_i^2(x)dx, \qquad \text{for } \; i = 0, 1, \ldots, n\,, \tag{4.2}$$

and the coefficients c_i can be determined directly. Also, we can increase n without disturbing the earlier coefficients.

Note, that any orthogonal set with n elements is linearly independent and hence, will always provide a basis for W, an n dimensional space, .

4.3.1 Generalisation of Least Squares

We can generalise the idea of least squares, using the inner product notation.

Suppose we define

$$\|f\|_2^2 = \langle f, f\rangle\,,$$

where $\langle ., . \rangle$ is some inner product (e.g., we considered the case $\langle f, g\rangle = \int_a^b fg dx$ in Chapter 1).

Then the least squares best approximation is the $\Psi(x)$ such that

$$\|f - \Psi\|_2$$

is minimised, i.e. we wish to minimise $\langle f - \Psi, f - \Psi\rangle$.

Writing $\Psi(x) = \sum_{i=0}^n c_i\phi_i(x)$, where $\phi_i \in P_n$ and form a basis for P_n and expressing orthogonality as $\langle \phi_i, \phi_j\rangle = 0$ for $i \neq j$, then choosing

$$c_i = \frac{\langle f(x), \phi_i(x)\rangle}{\langle \phi_i(x), \phi_i(x)\rangle}$$

(c.f. equation 4.2) guarantees that $\|f - \Psi\|_2 \leq \|f - p\|_2$ for all $p \in P_n$. In other words, Ψ is the best approximation to f out of P_n. *(See Tutorial sheet 4, question 1 for a derivation of this result).*

Example 4.3.1 *Find the least squares, straight line approximation to $x^{1/2}$ on $[0, 1]$. i.e., find the $\Psi(x) \in P_1$ that best fits $x^{1/2}$ on $[0, 1]$.*

First choose an orthogonal basis for P_1:

$$\phi_0(x) = 1 \qquad \text{and} \qquad \phi_1(x) = x - \frac{1}{2}\,.$$

These form an orthogonal basis for P_1 since

$$\int_0^1 \phi_0\phi_1 dx = \int_0^1 (x - \frac{1}{2})dx = \left[\frac{1}{2}x^2 - \frac{1}{2}x\right]_0^1 = \frac{1}{2} - \frac{1}{2} = 0\,.$$

Now construct $\Psi = c_0\phi_0 + c_1\phi_1 = c_0 + c_1(x - \frac{1}{2})$.

To find the Ψ which satisfies $||f - \Psi|| \leq ||f - p||$, we solve for the c_i as follows...

i=0:

$$c_0 = \frac{\langle f, \phi_0\rangle}{\langle \phi_0, \phi_0\rangle}$$

- $\langle f, \phi_0\rangle = \langle x^{1/2}, 1\rangle = \int_0^1 x^{1/2}dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3}$
- $\langle \phi_0, \phi_0\rangle = \langle 1, 1\rangle = \int_0^1 1dx = 1$

$$\Rightarrow c_0 = \frac{2}{3}$$

i=1:

$$c_1 = \frac{\langle f, \phi_1\rangle}{\langle \phi_1, \phi_1\rangle}$$

- $\langle f, \phi_1\rangle = \langle x^{1/2}, x - \frac{1}{2}\rangle = \int_0^1 x^{1/2}(x - \frac{1}{2})dx = \int_0^1 (x^{3/2} - \frac{1}{2}x^{1/2})dx = \left[\frac{2}{5}x^{2/5} - \frac{1}{3}x^{3/2}\right]_0^1 = \frac{1}{15}$
- $\langle \phi_1, \phi_1\rangle = \langle x - \frac{1}{2}, x - \frac{1}{2}\rangle = \int_0^1 (x - \frac{1}{2})^2 dx = \int_0^1 (x^2 - x + \frac{1}{4})dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x\right]_0^1 = \frac{1}{12}$

$$\Rightarrow c_1 = \frac{12}{15} = \frac{4}{5}$$

Hence, the least squares, straight line approximation to $x^{1/2}$ on $[0, 1]$ is $\Psi(x) = \frac{2}{3} + \frac{4}{5}\left(x - \frac{1}{2}\right) = \frac{4}{5}x + \frac{4}{15}$.

Example 4.3.2 *Show that a truncated Fourier Series is a least squares approximation of $f(x)$ for any $f(x)$ in the interval $[-\pi, \pi]$.*

Choose W to be the $2n + 1$ dimensional space of functions spanned by the basis

$$\phi_0 = 1, \phi_1 = \cos x, \phi_2 = \sin x, \phi_3 = \cos 2x, \phi_4 = \sin 2x, \ldots, \phi_{2n-1} = \cos nx, \phi_{2n} = \sin nx\,,$$

This basis forms an orthogonal set of functions:

e.g.

$$\int_{-\pi}^{\pi} \phi_0\phi_1 \mathrm{d}x = \int_{-\pi}^{\pi} \cos x \mathrm{d}x = [\sin x]_{-\pi}^{\pi} = 0\,, \qquad \text{etc.}, \ldots$$

Thus, a least squares approximation $\Psi(x)$ *of* $f(x)$ *can be written*

$$\Psi(x) = c_0 + c_1 \cos x + c_2 \sin x + \cdots + c_{2n-1} \cos nx + c_{2n} \sin nx \,,$$

with the c_i *given by*

$$c_0 = \frac{\langle f, \phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \,,$$
$$c_1 = \frac{\langle f, \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle} = \int_{-\pi}^{\pi} \cos x f(x) dx \Big/ \int_{-\pi}^{\pi} \cos^2 x dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x f(x) dx \,,$$

and so on.

The approximation Ψ *is the truncated Fourier series for* $f(x)$*. Hence, a Fourier series is an example of a Least Squares Approximation: a 'Best Approximation' in the least squares sense.*

Example 4.3.3 *Let* $\mathbf{x} = \{x_i\}\,,\ i = 1, ..., n$ *and* $\mathbf{y} = \{y_i\}\,,\ i = 1, ..., n$ *be the set of data points* (x_i, y_i)*. Find the least squares best straight line fit to these data points.*

We define the inner product in this case to be

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i \,,$$

Next we let

$$\boldsymbol{\Psi}(x) = \{c_1(x_i - \overline{x}) + c_0\}\,,\ i = 1, ..., n$$

with $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$*. Here* $\boldsymbol{\phi}_0(x) = 1\,,\ i = 1, ..., n$ *and* $\boldsymbol{\phi}_1(x) = \{x_i - \overline{x}\}\,,\ i, ..., n$*.*

Observe that

$$\langle \boldsymbol{\phi}_0(x), \boldsymbol{\phi}_1(x) \rangle = \sum_{i=1}^{n} (x_i - \overline{x}) \times 1 = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \overline{x} = n\overline{x} - n\overline{x} = 0 \,,$$

so $\boldsymbol{\phi}_0, \boldsymbol{\phi}_1$ *are an orthogonal set. Hence, if we calculate* c_0 *and* c_1 *as follows*

$$c_1 = \frac{\langle \mathbf{y}, \boldsymbol{\phi}_1 \rangle}{\langle \boldsymbol{\phi}_1, \boldsymbol{\phi}_1 \rangle} = \frac{\sum_{i=1}^{n} y_i (x_i - \overline{x})}{\sum_{i=1}^{n} (x_i - \overline{x})^2} \,,$$

and (using $\langle \boldsymbol{\phi}_0, \boldsymbol{\phi}_0 \rangle = \sum_{i=1}^{n} 1 = n$*)*

$$c_0 = \frac{\langle \mathbf{y}, \boldsymbol{\phi}_0 \rangle}{\langle \boldsymbol{\phi}_0, \boldsymbol{\phi}_0 \rangle} = \frac{\sum_{i=1}^{n} y_i}{n} \,.$$

then $\boldsymbol{\Psi}(x)$ *is the best linear fit (in a least squares sense) to the data points* (x_i, y_i)*.*

4.3.2 Approximations of Differing Degrees

Consider

$$\|f - \Psi\|_2 \le \|f - p(x)\|_2, \qquad \Psi, p \in P_n \,,$$

where $\Psi = \sum_{i=0}^{n} c_i\phi_i(x)\,,$ where $\phi_i(x)$ form an orthofonal basis for P_i.

Note, $p(x)$ may be ANY $p(x) \in P_n$, polynomials of degree at most n.
If we choose

$$p(x) = \sum_{i=0}^{n-1} c_i\phi_i(x)\,,$$

then $p(x) \in P_n$, and $p(x)$ is the best approximation to $f(x)$ of degree $n-1$ ($p(x) \in P_{n-1}$). Now from above we have

$$\|f - \Psi\|_2 \le \|f - \sum_{i=0}^{n-1} c_i\phi_i\|_2\,.$$

This means that the Least Squares Best approximation from P_n is at least as good as the Least Squares Best approximation from P_{n-1}. i.e. Adding more terms (higher degree basis functions) does not make the approximation worse - in fact, it will usually make it better.

4.4 Minimax

In the previous two sections, we have considered the best approximation in situations involving the $2-norm$. However, a best approximation in terms of the maximum (or infinity) norm:

$$\|f - p^*\|_\infty \le \|f - p\|_\infty, \qquad p \in P_n \,,$$

implies that we choose the polynomial that minimises the maximum error over $[a, b]$. This is a more natural way of thinking about 'Best Approximation'.

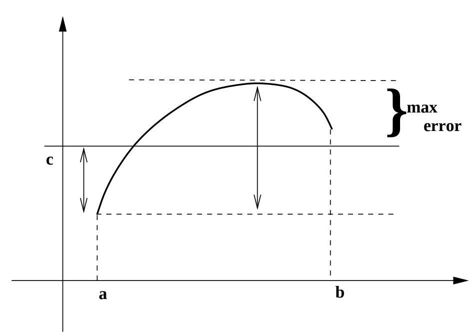
In such a situation, we call $p^*(x)$ the **minimax** approximation to $f(x)$ on $[a, b]$.

Example 4.4.1 *Find the best constant ($p* \in P_0$) approximation to $f(x)$ in the interval* $[a, b]$.
Let $c \in P_0$, thus we want to minimise $||f(x) - c||_\infty$:

$$\min_{\text{all } c}\left\{\max_{[a,b]}|f(x) - c|\right\},$$

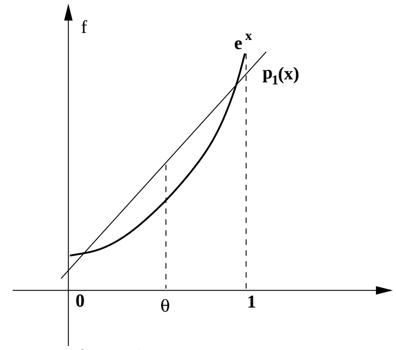
Clearly, the c that minimises this is

$$c = \frac{\max\{f\} + \min\{f\}}{2}\,.$$



Example 4.4.2 *Find the best straight line fit ($p* \in P_1$) to $f(x) = e^x$ in the interval* $[0, 1]$.

We want to find the straight line fit, hence we let $p* = mx + c$ *and we look to minimise*



$$||f(x) - p*||_\infty = ||e^x - (mx + c)||_\infty$$

i.e.,

$$\min_{\text{all } m,c} \left\{ \max_{[0,1]} |e^x - (mx + c)| \right\} .$$

Geometrically, the maximum occurs in three places, $x = 0$, $x = \theta$ *and* $x = 1$.

$$x = 0 : \qquad e^0 - (0 + c) = E \qquad \text{(i)}$$

$$x = \theta : \qquad e^\theta - (m\theta + c) = -E \qquad \text{(ii)}$$

$$x = 1 : \qquad e^1 - (m + c) = E \qquad \text{(iii)}$$

also, the error at $x = \theta$ *has a turning point, so that*

$$\frac{\partial}{\partial x}\left(e^x - (mx + c)\right)_{x=\theta} = 0 \Rightarrow e^\theta - m = 0 \qquad \Rightarrow \qquad m = e^\theta \qquad \Rightarrow \qquad \theta = \log_e m \,.$$

(i) and (iii) imply $1 - c = E = e - m - c$ *or,*

$$m = e - 1 \approx 1.7183 \qquad \Rightarrow \qquad \theta = \log_e(1.7183) \,.$$

(ii) and (iii) imply $e^\theta + e - m\theta - c - m - c = 0$ *or,*

$$c = \frac{1}{2}[m + e - m\theta - m] \approx 0.8941 \,.$$

Hence the minimax straight line is given by $1.7183x + 0.8941$.

As the above example illustrates, finding the minimax polynomial $p_n^*(x)$ for $n \geq 1$ is not a straight forward exercise. Also, note that the process involves the evaluation of the error, E in the above example.

4.4.1 Chebyshev Polynomials Revisited

Recall that the Chebyshev polynomials satistfy

$$\|\frac{1}{2^n}T_{n+1}(x)\|_\infty \leq \|q(x)\|_\infty \,,$$

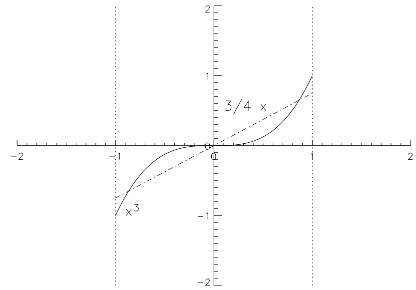
$\forall q(x) \in P_{n+1}$ such that $q(x) = x^{n+1} + \dots$.

In particular, if we consider $n = 2$, then

$$\left\| x^3 - \frac{3}{4}x \right\|_\infty \le \| x^3 + a_2 x^2 + a_1 x + a_0 \|_\infty ,$$

or

$$\left\| x^3 - \frac{3}{4}x \right\|_\infty \le \| x^3 - \left(-a_2 x^2 - a_1 x - a_0 \right) \|_\infty ,$$



$\forall$ constants a_0, a_1, a_2.

Hence

$$\left\| x^3 - \frac{3}{4}x \right\|_\infty \le \| x^3 - p_2(x) \|_\infty ,$$

$\forall p_2(x) \in P_2$.

This means the $p^*(x) \in P_2$ that is the minimax approximation to $f(x) = x^3$ in the interval $[-1, 1]$, i.e. the $p^*(x)$ that satisfies

$$\| x^3 - p_2^*(x) \|_\infty \le \| x^3 - p_2(x) \|_\infty .$$

is $p_2^*(x) = \frac{3}{4}x$.

From this example, we can see that the Chebyshev polynomial $T_{n+1}(x)$ can be used to quickly find the best polynomial of degree at most n (in the sense that the maximum error is minimised) to the function $f(x) = x^{n+1}$ in the interval $[-1, 1]$.

Finding the minimax approximation to $f(x) = x^{n+1}$ may see quite limited. However, in combination with the following results it can be very useful.

If $p_n^*(x)$ is the minimax approximation to $f(x)$ on $[a, b]$ from P_n then

1. $\alpha p_n^*(x)$ is the minimax approximation to $\alpha f(x)$ where $\alpha \in \mathbb{R}$,
   and

2. $p_n^*(x) + q_n(x)$ is the minimax approximation to $f(x) + q_n(x)$ where $q_n(x) \in P_n$.

(See Tutorial Sheet 8 for proofs and an example)

4.5 Equi-oscillation

From the above examples, we see that the error occurs several times.

- In Example 4.4.1: n=0 - maximum error occurred twice

- In Example 4.4.2: n=1 - maximum error occurred three times

• In Example 4.4.3: n=2 - maximum error occurred four times

In order to find the minimax approximation, we have found p_0, p_1 and p_2 such that the maximum error *equi-oscillates.*

Definition: A continuous function is said to **equi-oscillate** on n points of $[a, b]$ if there exist n points x_i

$$a \leq x_1 < x_2 < \cdots < x_n \leq b\,,$$

such that

$$|E(x_i)| = \max_{a \leq x \leq b} |E(x)|\,, \qquad i = 1, \ldots, n\,,$$

and

$$E(x_i) = -E(x_{i+1}), \qquad i = 1, \ldots, n-1\,.$$

Theorem:

For the function $\mathbf{f(x)}$, where $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$, and some $\mathbf{p_n(x)} \in \mathbf{P_n}$, suppose $\mathbf{f(x)} - \mathbf{p_n(x)}$ equi-oscillates on at least $(\mathbf{n}+\mathbf{2})$ points in $[\mathbf{a}, \mathbf{b}]$. Then $\mathbf{p_n(x)}$ is the minimax approximation for $\mathbf{f(x)}$.

(See Phillips & Taylor for a proof.)

The inverse of this theorem is also true: **if $\mathbf{p_n(x)}$ is the minimax polynomial of degree n, then $\mathbf{f(x)} - \mathbf{p_n(x)}$ equi-oscillates on at least $(\mathbf{n}+\mathbf{2})$ points.**

The property of equi-oscillation characterises the minimax approximation.

Example 4.5.1 *Construct the minimax, straight line approximation to* $x^{1/2}$ *on* $[0, 1]$.

So we wish to find $p_1(x) = mx + c$ *such that*

$$\max_{[0,1]} \left| x^{1/2} - (mx + c) \right|$$

is minimised.

From the above theorem we know the maximum must occur in $n + 2 = 3$ *places,* $x = 0$, $x = \theta$ *and* $x = 1$.

$$x = 0: \qquad 0 - (0 + c) = -E \qquad \text{(i)}$$

$$x = \theta: \qquad \theta^{1/2} - (m\theta + c) = E \qquad \text{(ii)}$$

$$x = 1: \qquad 1 - (m + c) = -E \qquad \text{(iii)}$$

Also, the error at $x = \theta$ has a turning point:

$$
\begin{aligned}
&\Rightarrow \quad \frac{\partial}{\partial x}\left(x^{1/2} - (mx + c)\right)_{x=\theta} = 0 \\
&\Rightarrow \quad \left(\frac{1}{2}x^{-1/2} - m\right)_{x=\theta} = 0 \\
&\Rightarrow \quad \frac{1}{2}\theta^{-1/2} - m = 0 \\
&\Rightarrow \quad \theta = \frac{1}{4m^2}\,.
\end{aligned}
$$

Combining (i) and (iii): $-c = 1 - m - c \Rightarrow m = 1$ *Combining (ii) and (iii):*

$$
\begin{aligned}
&\Rightarrow \quad \theta^{1/2} - (m\theta + c) + 1 - (m + c) = 0 \\
&\Rightarrow \quad \frac{1}{2m} - \frac{1}{4m} + 1 - m - 2c = 0 \\
&\Rightarrow \quad \frac{1}{2} - \frac{1}{4} + 1 - 1 - 2c = 0 \\
&\Rightarrow \quad c = \frac{1}{8}\,.
\end{aligned}
$$

Hence the minimax straight line approximation to $x^{1/2}$ is given by $x + \frac{1}{8}$.

On the other hand, the least squares, straight line approximation was $\frac{4}{5}x + \frac{4}{15}$, making it clear that different norms lead to different approximations!

4.6 Chebyshev Series Again

The property of equi-oscillation characterises the minimax approximation. Suppose we could produce the following series expansion,

$$f(x) = \sum_{i=0}^{\infty} a_i T_i(x)$$

for $f(x)$ defined on $[-1, 1]$. This is called a **Chebyshev series**.

Not such a crazy idea! Put $x = \cos\theta$, then

$$f(\cos\theta) = \sum_{i=0}^{\infty} a_i T_i(\cos\theta) = \sum_{i=0}^{\infty} a_i \cos(i\theta), \qquad 0 \le \theta \le \pi\,,$$

which is just the Fourier cosine series for the function $f(\cos\theta)$.

Hence, it is a series we could evaluate (using numerical integration if necessary).

Now, suppose the series converges rapidly so that,

$$|a_{n+1}| \gg |a_{n+2}| \gg |a_{n+3}| \gg \ldots$$

so a few terms are a good approximation of the function.

Let $\Psi(x) = \sum_{i=0}^{n} a_i T_i(x)$ then

$$\begin{aligned} f(x) - \Psi(x) &= a_{n+1}T_{n+1}(x) + a_{n+2}T_{n+2}(x) + \ldots \\ &\backsimeq a_{n+1}T_{n+1}(x)\,, \end{aligned}$$

or, the error is dominated by the leading term $a_{n+1}T_{n+1}(x)$. Now $T_{n+1}(x)$ equi-oscillates $(n+2)$ times on $[-1,1]$.

If $f(x) - \Psi(x) = a_{n+1}T_{n+1}(x)$, then $\Psi(x)$ would be the **minimax** polynomial of degree n to $f(x)$. Since

$$f(x) - \Psi(x) \backsimeq a_{n+1}T_{n+1}(x)\,,$$

$\Psi(x)$ is not the minimax but is a polynomial that is 'close' to the minimax, as long as $a_{n+2}, a_{n+3}, \ldots$ are small compared to a_{n+1}.

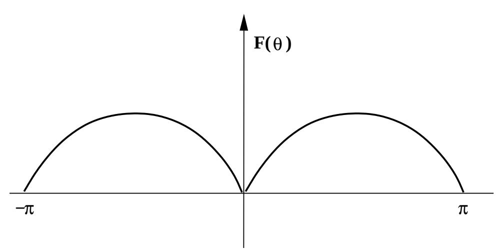
The actual error almost equi-oscillates on $(n+2)$ points.

Example 4.6.1: *Find the minimax quadratic approximation to* $f(x) = (1-x^2)^{1/2}$ *in the interval* $[-1,1]$.

First, we note that if $x = \cos\theta$ *then* $f(cos\theta) = (1-\cos^2\theta)^{1/2} = \sin\theta$ *and the interval* $x \in [-1,1]$ *becomes* $\theta \in [0,\pi]$.

The Fourier cosine series for $\sin\theta$ *on* $[0,\pi]$ *is given by*

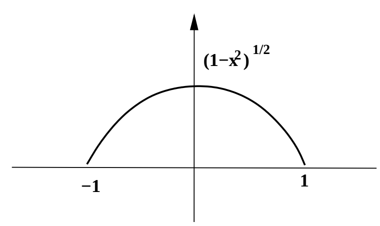
$$\sin\theta = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2\theta}{3} + \frac{\cos 4\theta}{15} + \frac{\cos 6\theta}{35} + \ldots\right]$$



So with $x = \cos\theta$, *we have*

$$(1-x^2)^{1/2} = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{T_2(x)}{3} + \frac{T_4(x)}{15} + \frac{T_6(x)}{35} + \ldots\right],$$

where $-1 \le x \le 1$.



Thus let use consider the quadratic

$$\begin{aligned} p_2(x) &= \frac{2}{\pi} - \frac{4}{\pi}\frac{T_2(x)}{3} = \frac{2}{\pi} - \frac{4}{3\pi}(2x^2-1) \\ &= \frac{2}{3\pi}(3 - 2(2x^2-1)) = \frac{2}{3\pi}(5-4x^2)\,. \end{aligned}$$

The error

$$f(x) - p_2(x) \approx -\frac{4}{\pi}\frac{T_4(x)}{15}\,,$$

which oscillates $4+1=5$ times in [-1,1]. At least 4 equi-oscillation points are required for $p_2(x)$ to be the minimax approximation of $(1-x^2)^{1/2}$, so we need to see whether the above oscillation points are of equal amplitude.

$T_4(x)$ has extreme values when $8x^4 - 8x^2 + 1 = \pm 1$, i.e. at

$$x = 0\,,\ x = 1\,,\ x = -1\,,\ x = 1/\sqrt{2}\ \ \text{and}\ \ x = -1/\sqrt{2}\,.$$

	$(1-x^2)^{1/2}$	$p_2(x)$	*error*
$x = 0$	1	$10/3\pi$	-0.0610
$x = \pm 1/\sqrt{2}$	$1/\sqrt{2}$	$2/\pi$	0.0705
$x = \pm 1$	0	$2/3\pi$	-0.2122

So the error oscillates but not equally. Hence, $p_2(x)$ is not quite the minimax approximation to $f(x) = (1-x^2)^{1/2}$, but it is a good first approximation.

The true minimax quadratic to $(1-x^2)^{1/2}$ is actually $\left(\frac{9}{8} - x^2\right) = (1.125 - x^2)$, and thus our estimate of $(1.061 - 0.8488x^2)$ is not bad.

4.7 Economisation of a Power Series

Another way of exploiting the properties of Chebyshev polynomials is possible for functions $f(x)$ for which a power series exists.

Consider the function $f(x)$ which equals the power series

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \ .$$

Let us assume that we are interested in approximating $f(x)$ with a polynomial of degree m.

One such approximation is

$$f(x) = \sum_{n=1}^{m} a_n x^n + R_m \ ,$$

which has error R_m. Can we get a better approximation of degree m than this?

Yes! A better approximation may be found by finding a function $p_m(x)$ such that $f(x) - p_m(x)$ equi-oscillates at least $m+2$ times in the given interval.

Consider the truncated series of degree $m+1$

$$f(x) = \sum_{n=1}^{m} a_n x^n + a_{m+1}x^{m+1} + R_{m+1} \ .$$

The Chebyshev polynomial of degree $m+1$, equi-oscillates $m+2$ times, and equals

$$T_{m+1}(x) = 2^m x^{m+1} + t_{m-1}(x) \ ,$$

where t_{m-1} are the terms in the Chebyshev polynomial of degree at most $m-1$. Hence, we can write

$$x^{m+1} = \frac{1}{2^m}\left(T_{m+1}(x) - t_{m-1}(x)\right) .$$

Substituting for x^{m+1} in our expression for $f(x)$ we get

$$f(x) = \sum_{n=1}^{m} a_n x^n + \frac{a_{m+1}}{2^m}\left(T_{m+1}(x) - t_{m-1}(x)\right) + R_{m+1} .$$

Re-arranging we find a polynomial of degree at most m,

$$p_m(x) = \sum_{n=1}^{m} a_n x^n - \frac{a_{m+1}}{2^m} t_{m-1}(x) .$$

This polynomial will be a pretty good approximation to $f(x)$ since

$$f(x) - p_m(x) = \frac{a_{m+1}}{2^m} T_{m+1}(x) + R_{m+1} ,$$

which oscillates $m+2$ times almost equally provided R_{m+1} is small.

Although $p_m(x)$ is not the minimax approximation to $f(x)$ it is close and the error

$$\frac{a_{m+1}}{2^m} T_{m+1}(x) + R_{m+1} \leq \frac{a_{m+1}}{2^m} + R_{m+1} ,$$

since $|T_{m+1}(x)| \leq 1$, is generally a lot less than the error R_m for the truncated power series of degree m.

This process is called the *Economisation of a power series.*

Example 4.7.1: *The Taylor expansion of* $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + R_7 ,$$

where

$$R_7 = \frac{x^7}{7!}\frac{d^7}{dx^7}(\sin x)_{x=\theta} = \frac{x^7}{7!}(-\cos\theta) .$$

For $x \in [-1, 1]$, $|R_7| \leq \frac{1}{7!} \approx 0.0002$.

However,

$$\sin x = x - \frac{x^3}{3!} + R_5 ,$$

where

$$R_5 = \frac{x^5}{5!}\frac{d^5}{dx^5}(\sin x)_{x=\theta} = \frac{x^5}{5!}(\cos\theta) ,$$

so $|R_5| \leq \frac{1}{5!} \approx 0.0083$*. The extra term makes a big difference!*

Now suppose we express x^5 *in terms of Chebyshev polynomials,*

$$T_5(x) = 16x^5 - 20x^3 + 5x ,$$

so

$$x^5 = \frac{T_5(x) + 20x^3 - 5x}{16} \, .$$

Then

$$\begin{aligned}
\sin x &= x - \frac{x^3}{6} + \frac{1}{5!}\left(\frac{T_5(x) + 20x^3 - 5x}{16}\right) + R_7 \\
&= x\left(1 - \frac{1}{16 \times 4!}\right) - \frac{x^3}{6}\left(1 - \frac{1}{16}\right) + \frac{1}{16 \times 5!}T_5(x) + R_7 \, .
\end{aligned}$$

Now $|T_5(x)| \le 1$ *for* $x \in [-1, 1]$ *so if we ignore the term in* $T_5(x)$ *we obtain*

$$\sin x = x\left(1 - \frac{1}{16 \times 4!}\right) - \frac{x^3}{6} \times \frac{15}{16} + \text{Error}$$

where

$$\begin{aligned}
|\text{Error}| &\le |R_7| + \frac{1}{16 \times 5!}|T_5(x)| \, , \\
&\le 0.0002 + \frac{1}{16 \times 120} = 0.0002 + \frac{1}{1920} \\
&\le 0.0002 + 0.00052 \simeq 0.0007 \, .
\end{aligned}$$

This new cubic has maximum error of about 0.0007, compared with 0.0083 for $x - \frac{x^3}{6}$.