Chapter 4

Best Approximation

4.1 The General Case

In the previous chapter, we have seen how an interpolating polynomial can be used as an approximation to a given function. We now want to find the best approximation to a given function.

This fundamental problem in Approximation Theory can be stated in very general terms. Let $V$ be a Normed Linear Space and $W$ a finite-dimensional subspace of $V$, then for a given $v \in V$, find $w^* \in W$ such that

$$
\|v - w^*\| \leq \|v - w\|
$$

for all $w \in W$. Here $w^*$ is called the Best Approximation to $v$ out of the subspace $W$. Note that the definition of $V$ defines the particular norm to be used and, when using that norm, $w^*$ is the vector that is closest to $v$ out of all possible vectors in $W$. In general, different norms lead to different approximations.

In the context of Numerical Analysis, $V$ is usually the set of continuous functions on some interval $[a, b]$, with some selected norm, and $W$ is usually the space of polynomials $P_n$. The requirement that $W$ is finite-dimensional ensures that we have a basis for $W$.

Least Squares Problem

Let $f(x)$ be a given particular continuous function. Using the 2-norm

$$
\|f(x)\|_2 = \left( \int_a^b f^2(x)dx \right)^{1/2}
$$

find $p^*(x)$ such that

$$
\|f(x) - p^*(x)\|_2 \leq \|f(x) - p(x)\|_2,
$$
for all \( p(x) \in P_n \), polynomials of degree at most \( n \), and \( x \in [a, b] \).

This is known as the **Least Squares Problem**. Best approximations with respect to the 2-norm are called **least squares approximations**.

### 4.2 Least Squares Approximation

In the above problem, how do we find \( p^*(x) \)? The procedure is the same, regardless of the subspace used.

So let \( W \) be any finite-dimensional subspace of dimension \((n + 1)\), with basis vectors

\[
\phi_0(x), \phi_1(x), \ldots \text{ and } \phi_n(x).
\]

Therefore, any member of \( W \) can be expressed as

\[
\Psi(x) = \sum_{i=0}^{n} c_i \phi_i(x),
\]

where \( c_i \in \mathbb{R} \). The problem is to find \( c_i \) such that \( \|f - \Psi\|_2 \) is **minimised**.

Define

\[
E(c_0, c_1, \ldots, c_n) = \int_a^b (f(x) - \Psi(x))^2 dx.
\]

We require the minimum of \( E(c_0, c_1, \ldots, c_n) \) over all values \( c_0, c_1, \ldots, c_n \). A necessary condition for \( E \) to have a minimum is:

\[
\frac{\partial E}{\partial c_i} = 0 = -2 \int_a^b (f - \Psi) \frac{\partial \Psi}{\partial c_i} dx,
\]

\[
= -2 \int_a^b (f - \Psi) \phi_i(x) dx.
\]

This implies,

\[
\int_a^b f(x) \phi_i(x) dx = \int_a^b \Psi \phi_i(x) dx,
\]

or

\[
\int_a^b f(x) \phi_i(x) dx = \int_a^b \sum_{j=0}^{n} c_j \phi_j(x) \phi_i(x) dx.
\]

Hence, the \( c_i \) that minimise \( \|f(x) - \Psi(x)\|_2 \) satisfy the system of equations given by

\[
\int_a^b f(x) \phi_i(x) dx = \sum_{j=0}^{n} c_j \int_a^b \phi_j(x) \phi_i(x) dx, \quad \text{for } i = 0, 1, \ldots, n,
\]

a total of \((n + 1)\) equations in \((n + 1)\) unknowns \( c_0, c_1, \ldots, c_n \).

These equations are often called the **Normal Equations**.
Example 4.2.1 Using the Normal Equations (4.1) find the \( p(x) \in P_n \) the best fits, in a least squares sense, a general continuous function \( f(x) \) in the interval \([0, 1]\).

i.e. find \( p^*(x) \) such that

\[
\| f(x) - p^*(x) \|_2 \leq \| f(x) - p(x) \|_2,
\]

for all \( p(x) \in P_n \), polynomials of degree at most \( n \), and \( x \in [0, 1] \).

Take the basis for \( P_n \) as

\[
\phi_0 = 1, \phi_1 = x, \phi_2 = x^2, \ldots, \phi_n = x^n.
\]

Then

\[
\int_0^1 f(x)x^i dx = \sum_{j=0}^n c_j \int_0^1 x^j x^i dx
\]

\[
= \sum_{j=0}^n c_j \int_0^1 x^{i+j} dx
\]

\[
= \sum_{j=0}^n c_j \left[ \frac{x^{i+j+1}}{i+j+1} \right]_0^1
\]

\[
= \sum_{j=0}^n c_j \frac{1}{i+j+1}.
\]

Or, writing them out:

\[
i = 0: \quad \int_0^1 f(x) dx = c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \cdots + \frac{c_n}{n+1}
\]

\[
i = 1: \quad \int_0^1 x f(x) dx = \frac{c_0}{2} + \frac{c_1}{3} + \frac{c_2}{4} + \cdots + \frac{c_n}{n+2}
\]

\[
\cdots
\]

\[
i = n: \quad \int_0^1 x^n f(x) dx = \frac{c_0}{n+1} + \frac{c_1}{n+2} + \cdots + \frac{c_n}{2n+1}
\]

Or, in matrix form:

\[
\begin{bmatrix}
1 & 1/2 & \cdots & 1/n+1 \\
1/2 & 1/3 & \cdots & 1/n+2 \\
\vdots & \vdots & \ddots & \vdots \\
1/n+1 & 1/n+2 & \cdots & 1/2n+1
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_n
\end{bmatrix}
= 
\begin{bmatrix}
\int_0^1 f(x) dx \\
\int_0^1 x f(x) dx \\
\vdots \\
\int_0^1 x^n f(x) dx
\end{bmatrix}
\]

Does anything look familiar? A system \( HA = f \) where \( H \) is the Hilbert matrix. This is seriously bad news - this system is famously ILL-CONDITIONED! We will have to find a better way to find \( p^* \).
4.3 Orthogonal Functions

In general, it will be hard to solve the Normal Equations, as the Hilbert matrix is ill-conditioned. The previous example is an example of what not to do!

Instead, using the same approach as before choose (if possible) an orthogonal basis \( \phi_i(x) \) such that
\[
\int_a^b \phi_i(x) \phi_j(x) \, dx = 0, \quad i \neq j.
\]

In this case, the Normal Equations (4.1) reduce to
\[
\int_a^b f(x) \phi_i(x) \, dx = c_i \int_a^b \phi_i^2(x) \, dx, \quad \text{for } i = 0, 1, \ldots, n, \tag{4.2}
\]
and the coefficients \( c_i \) can be determined directly. Also, we can increase \( n \) without disturbing the earlier coefficients.

Note, that any orthogonal set with \( n \) elements is linearly independent and hence, will always provide a basis for \( W \), an \( n \) dimensional space.

4.3.1 Generalisation of Least Squares

We can generalise the idea of least squares, using the inner product notation.

Suppose we define
\[
\| f \|_2^2 = \langle f, f \rangle,
\]
where \( \langle \cdot, \cdot \rangle \) is some inner product (e.g., we considered the case \( \langle f, g \rangle = \int_a^b fg \, dx \) in Chapter 1).

Then the least squares best approximation is the \( \Psi(x) \) such that
\[
\| f - \Psi \|_2
\]
is minimised, i.e. we wish to minimise \( \langle f - \Psi, f - \Psi \rangle \).

Writing \( \Psi(x) = \sum_{i=0}^n c_i \phi_i(x) \), where \( \phi_i \in P_n \) and form a basis for \( P_n \) and expressing orthogonality as \( \langle \phi_i, \phi_j \rangle = 0 \) for \( i \neq j \), then choosing
\[
c_i = \frac{\langle f(x), \phi_i(x) \rangle}{\langle \phi_i(x), \phi_i(x) \rangle}
\]
(c.f. equation 4.2) guarantees that \( \| f - \Psi \|_2 \leq \| f - p \|_2 \) for all \( p \in P_n \). In other words, \( \Psi \) is the best approximation to \( f \) out of \( P_n \). (See Tutorial sheet 4, question 1 for a derivation of this result).

Example 4.3.1 Find the least squares, straight line approximation to \( x^{1/2} \) on \([0, 1]\). i.e., find the \( \Psi(x) \in P_1 \) that best fits \( x^{1/2} \) on \([0, 1] \).
First choose an orthogonal basis for \( P_1 \):

\[
\phi_0(x) = 1 \quad \text{and} \quad \phi_1(x) = x - \frac{1}{2}.
\]

These form an orthogonal basis for \( P_1 \) since

\[
\int_0^1 \phi_0 \phi_1 \, dx = \int_0^1 (x - \frac{1}{2}) \, dx = \left[ \frac{1}{2} x^2 - \frac{1}{2} x \right]_0^1 = \frac{1}{2} - \frac{1}{2} = 0.
\]

Now construct \( \Psi = c_0 \phi_0 + c_1 \phi_1 = c_0 + c_1(x - \frac{1}{2}) \).

To find the \( \Psi \) which satisfies \( \| f - \Psi \| \leq \| f - p \| \), we solve for the \( c_i \) as follows...

\[ i=0: \]

\[
c_0 = \frac{\langle f, \phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle}
\]

\[
\bullet \, \langle f, \phi_0 \rangle = \langle x^{1/2}, 1 \rangle = \int_0^1 x^{1/2} \, dx = \left[ \frac{2}{3} x^{3/2} \right]_0^1 = \frac{2}{3}
\]

\[
\bullet \, \langle \phi_0, \phi_0 \rangle = \langle 1, 1 \rangle = \int_0^1 1 \, dx = 1
\]

\[ \Rightarrow c_0 = \frac{2}{3} \]

\[ i=1: \]

\[
c_1 = \frac{\langle f, \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle}
\]

\[
\bullet \, \langle f, \phi_1 \rangle = \langle x^{1/2}, x - \frac{1}{2} \rangle = \int_0^1 x^{1/2}(x - \frac{1}{2}) \, dx = \int_0^1 (x^{3/2} - \frac{3}{2} x^{1/2}) \, dx = \left[ \frac{2}{3} x^{3/2} - \frac{3}{2} x^{1/2} \right]_0^1 = \frac{1}{15}
\]

\[
\bullet \, \langle \phi_1, \phi_1 \rangle = \langle x - \frac{1}{2}, x - \frac{1}{2} \rangle = \int_0^1 (x - \frac{1}{2})^2 \, dx = \int_0^1 (x^2 - x + \frac{1}{4}) \, dx = \left[ \frac{1}{3} x^3 - \frac{3}{2} x^2 + \frac{x}{4} \right]_0^1 = \frac{1}{12}
\]

\[ \Rightarrow c_1 = \frac{12}{15} = \frac{4}{5} \]

Hence, the least squares, straight line approximation to \( x^{1/2} \) on \( [0, 1] \) is \( \Psi(x) = \frac{2}{3} + \frac{4}{5} \left( x - \frac{1}{2} \right) = \frac{4}{5} x + \frac{1}{5} \).

Example 4.3.2 Show that a truncated Fourier Series is a least squares approximation of \( f(x) \) for any \( f(x) \) in the interval \([ -\pi, \pi] \).

Choose \( W \) to be the \( 2n + 1 \) dimensional space of functions spanned by the basis

\[
\phi_0 = 1, \phi_1 = \cos x, \phi_2 = \sin x, \phi_3 = \cos 2x, \phi_4 = \sin 2x, \ldots, \phi_{2n-1} = \cos nx, \phi_{2n} = \sin nx,
\]

This basis forms an orthogonal set of functions:

\[ e.g. \]

\[
\int_{-\pi}^{\pi} \phi_0 \phi_1 \, dx = \int_{-\pi}^{\pi} \cos x \, dx = [\sin x]_{-\pi}^{\pi} = 0, \quad \text{etc.,} \ldots
\]
Thus, a least squares approximation $\Psi(x)$ of $f(x)$ can be written

$$\Psi(x) = c_0 + c_1 \cos x + c_2 \sin x + \cdots + c_{2n-1} \cos nx + c_{2n} \sin nx,$$

with the $c_i$ given by

$$c_0 = \frac{\langle f, \phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx,$$

$$c_1 = \frac{\langle f, \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle} = \int_{-\pi}^{\pi} \cos x f(x) dx / \int_{-\pi}^{\pi} \cos^2 x dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x f(x) dx,$$

and so on.

The approximation $\Psi$ is the truncated Fourier series for $f(x)$. Hence, a Fourier series is an example of a Least Squares Approximation: a ‘Best Approximation’ in the least squares sense.

**Example 4.3.3** Let $x = \{x_i\}, i = 1, \ldots, n$ and $y = \{y_i\}, i = 1, \ldots, n$ be the set of data points $(x_i, y_i)$. Find the least squares best straight line fit to these data points.

We define the inner product in this case to be

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i,$$

Next we let

$$\Psi(x) = \{c_1(x_i - \overline{x}) + c_0\}, i = 1, \ldots, n$$

with $\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$. Here $\phi_0(x) = 1$, $i = 1, \ldots, n$ and $\phi_1(x) = \{x_i - \overline{x}\}, i, \ldots, n$.

Observe that

$$\langle \phi_0(x), \phi_1(x) \rangle = \sum_{i=1}^{n} (x_i - \overline{x}) \times 1 = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \overline{x} = n\overline{x} - n\overline{x} = 0,$$

so $\phi_0, \phi_1$ are an orthogonal set. Hence, if we calculate $c_0$ and $c_1$ as follows

$$c_1 = \frac{\langle y, \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle} = \frac{\sum_{i=1}^{n} y_i (x_i - \overline{x})}{\sum_{i=1}^{n} (x_i - \overline{x})^2},$$

and (using $\langle \phi_0, \phi_0 \rangle = \sum_{i=1}^{n} 1 = n$)

$$c_0 = \frac{\langle y, \phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle} = \frac{\sum_{i=1}^{n} y_i}{n},$$

then $\Psi(x)$ is the best linear fit (in a least squares sense) to the data points $(x_i, y_i)$. 74
4.3.2 Approximations of Differing Degrees

Consider
\[ \| f - \Psi \|_2 \leq \| f - p(x) \|_2, \quad \Psi, p \in P_n, \]
where \( \Psi = \sum_{i=0}^{n} c_i \phi_i(x) \), where \( \phi_i(x) \) form an orthogonal basis for \( P_1 \).

Note, \( p(x) \) may be ANY \( p(x) \in P_n \), polynomials of degree at most \( n \).

If we choose \( p(x) = \sum_{i=0}^{n-1} c_i \phi_i(x) \), then \( p(x) \in P_n \), and \( p(x) \) is the best approximation to \( f(x) \) of degree \( n - 1 \) (\( p(x) \in P_{n-1} \)). Now from above we have
\[ \| f - \Psi \|_2 \leq \| f - \sum_{i=0}^{n-1} c_i \phi_i \|_2. \]
This means that the Least Squares Best approximation from \( P_n \) is at least as good as the Least Squares Best approximation from \( P_{n-1} \). i.e. Adding more terms (higher degree basis functions) does not make the approximation worse - in fact, it will usually make it better.

4.4 Minimax

In the previous two sections, we have considered the best approximation in situations involving the \( 2 - norm \). However, a best approximation in terms of the maximum (or infinity) norm:
\[ \| f - p^* \|_\infty \leq \| f - p \|_\infty, \quad p \in P_n, \]
implies that we choose the polynomial that minimises the maximum error over \( [a, b] \). This is a more natural way of thinking about ‘Best Approximation’.

In such a situation, we call \( p^*(x) \) the \textbf{minimax} approximation to \( f(x) \) on \( [a, b] \).

**Example 4.4.1** Find the best constant (\( p^* \in P_0 \)) approximation to \( f(x) \) in the interval \( [a, b] \).

Let \( c \in P_0 \), thus we want to minimise \( \| f(x) - c \|_\infty \):
\[ \min_{c} \left\{ \max_{[a,b]} |f(x) - c| \right\}, \]
Clearly, the \( c \) that minimises this is
\[ c = \frac{\max\{f\} + \min\{f\}}{2}. \]

**Example 4.4.2** Find the best straight line fit (\( p^* \in P_1 \)) to \( f(x) = e^x \) in the interval \([0, 1]\).
We want to find the straight line fit, hence we let $p^* = mx + c$ and we look to minimise
\[ ||f(x) - p^*||_\infty = ||e^x - (mx + c)||_\infty \]
i.e.,
\[ \min_{m,c} \left\{ \max_{[0,1]} |e^x - (mx + c)| \right\} . \]

Geometrically, the maximum occurs in three places, $x = 0$, $x = \theta$ and $x = 1$.

\begin{align*}
x = 0 : & \quad e^0 - (0 + c) = E & \text{(i)} \\
x = \theta : & \quad e^\theta - (m\theta + c) = -E & \text{(ii)} \\
x = 1 : & \quad e^1 - (m + c) = E & \text{(iii)}
\end{align*}

also, the error at $x = \theta$ has a turning point, so that
\[ \frac{\partial}{\partial x} (e^x - (mx + c))_{x=\theta} = 0 \Rightarrow e^\theta - m = 0 \quad \Rightarrow \quad m = e^\theta \quad \Rightarrow \quad \theta = \log_e m . \]

(i) and (iii) imply $1 - c = E = e - m - c$ or,
\[ m = e - 1 \approx 1.7183 \quad \Rightarrow \quad \theta = \log_e(1.7183) . \]

(ii) and (iii) imply $e^\theta + e - m\theta - c - m - c = 0$ or,
\[ c = \frac{1}{2} [m + e - m\theta - m] \approx 0.8941 . \]

Hence the minimax straight line is given by $1.7183x + 0.8941$.

As the above example illustrates, finding the minimax polynomial $p_n^*(x)$ for $n \geq 1$ is not a straightforward exercise. Also, note that the process involves the evaluation of the error, $E$ in the above example.

### 4.4.1 Chebyshev Polynomials Revisited

Recall that the Chebyshev polynomials satisfy
\[ \| \frac{1}{2^n} T_{n+1}(x) \|_\infty \leq \| q(x) \|_\infty , \]
\[ \forall q(x) \in P_{n+1} \text{ such that } q(x) = x^{n+1} + \ldots . \]
In particular, if we consider \( n = 2 \), then
\[
\left\| x^3 - \frac{3}{4}x \right\|_\infty \leq \left\| x^3 + a_2x^2 + a_1x + a_0 \right\|_\infty ,
\]
or
\[
\left\| x^3 - \frac{3}{4}x \right\|_\infty \leq \left\| x^3 - (-a_2x^2 - a_1x - a_0) \right\|_\infty ,
\]
\( \forall \) constants \( a_0, a_1, a_2 \).

Hence
\[
\left\| x^3 - \frac{3}{4}x \right\|_\infty \leq \left\| x^3 - p_2(x) \right\|_\infty ,
\]
\( \forall p_2(x) \in P_2 \).

This means the \( p^*(x) \in P_2 \) that is the minimax approximation to \( f(x) = x^3 \) in the interval \([ -1, 1] \), i.e. the \( p^*(x) \) that satisfies
\[
\left\| x^3 - p^*_2(x) \right\|_\infty \leq \left\| x^3 - p_2(x) \right\|_\infty .
\]
is \( p^*_2(x) = \frac{3}{4}x \).

From this example, we can see that the Chebyshev polynomial \( T_{n+1}(x) \) can be used to quickly find the best polynomial of degree at most \( n \) (in the sense that the maximum error is minimised) to the function \( f(x) = x^{n+1} \) in the interval \([-1, 1] \).

Finding the minimax approximation to \( f(x) = x^{n+1} \) may see quite limited. However, in combination with the following results it can be very useful.

If \( p^*_n(x) \) is the minimax approximation to \( f(x) \) on \([ a, b] \) from \( P_n \) then

1. \( \alpha p^*_n(x) \) is the minimax approximation to \( \alpha f(x) \) where \( \alpha \in \mathbb{R} \), and

2. \( p^*_n(x) + q_n(x) \) is the minimax approximation to \( f(x) + q_n(x) \) where \( q_n(x) \in P_n \).

(See Tutorial Sheet 8 for proofs and an example)

### 4.5 Equi-oscillation

From the above examples, we see that the error occurs several times.

- In Example 4.4.1: \( n=0 \) - maximum error occurred twice
- In Example 4.4.2: \( n=1 \) - maximum error occurred three times
In Example 4.4.3: n=2 - maximum error occurred four times

In order to find the minimax approximation, we have found \( p_0, p_1 \) and \( p_2 \) such that the maximum error equi-oscillates.

**Definition:** A continuous function is said to **equi-oscillate** on \( n \) points of \([a,b]\) if there exist \( n \) points \( x_i \)

\[
a \leq x_1 < x_2 < \cdots < x_n \leq b,
\]

such that

\[
|E(x_i)| = \max_{a \leq x \leq b} |E(x)|, \quad i = 1, \ldots, n,
\]

and

\[
E(x_i) = -E(x_{i+1}), \quad i = 1, \ldots, n-1.
\]

**Theorem:**

For the function \( f(x) \), where \( x \in [a,b] \), and some \( p_n(x) \in P_n \), suppose \( f(x) - p_n(x) \) equi-oscillates on at least \( (n+2) \) points in \([a,b]\). Then \( p_n(x) \) is the **minimax** approximation for \( f(x) \).

(See Phillips & Taylor for a proof.)

The inverse of this theorem is also true: if \( p_n(x) \) is the minimax polynomial of degree \( n \), then \( f(x) - p_n(x) \) equi-oscillates on at least \( (n+2) \) points.

The property of equi-oscillation characterises the minimax approximation.

**Example 4.5.1** Construct the minimax, straight line approximation to \( x^{1/2} \) on \([0,1]\).

So we wish to find \( p_1(x) = mx + c \) such that

\[
\max_{[0,1]} |x^{1/2} - (mx + c)|
\]

is minimised.

From the above theorem we know the maximum must occur in \( n+2 = 3 \) places, \( x = 0, x = \theta \) and \( x = 1 \).

\[
\begin{align*}
    x = 0 : & \quad 0 - (0 + c) = -E \quad \text{(i)} \\
    x = \theta : & \quad \theta^{1/2} - (m\theta + c) = E \quad \text{(ii)} \\
    x = 1 : & \quad 1 - (m + c) = -E \quad \text{(iii)}
\end{align*}
\]
Also, the error at $x = \theta$ has a turning point:

\[
\Rightarrow \frac{\partial}{\partial x} \left( x^{1/2} - (mx + c) \right)_{x=\theta} = 0
\]

\[
\Rightarrow \left( \frac{1}{2} x^{-1/2} - m \right)_{x=\theta} = 0
\]

\[
\Rightarrow \frac{1}{2} \theta^{-1/2} - m = 0
\]

\[
\Rightarrow \theta = \frac{1}{4m^2}.
\]

Combining (i) and (iii): $-c = 1 - m - c \Rightarrow m = 1$
Combining (ii) and (iii):

\[
\Rightarrow \theta^{1/2} - (m\theta + c) + 1 - (m + c) = 0
\]

\[
\Rightarrow \frac{1}{2m} - \frac{1}{4m} + 1 - m - 2c = 0
\]

\[
\Rightarrow \frac{1}{2} - \frac{1}{4} + 1 - 1 - 2c = 0
\]

\[
\Rightarrow c = \frac{1}{8}.
\]

Hence the minimax straight line approximation to $x^{1/2}$ is given by $x + \frac{1}{8}$.

On the other hand, the least squares, straight line approximation was $\frac{4}{5}x + \frac{4}{15}$, making it clear that different norms lead to different approximations!

### 4.6 Chebyshev Series Again

The property of equi-oscillation characterises the minimax approximation. Suppose we could produce the following series expansion,

\[
f(x) = \sum_{i=0}^{\infty} a_i T_i(x)
\]

for $f(x)$ defined on $[-1, 1]$. This is called a Chebyshev series.

Not such a crazy idea! Put $x = \cos \theta$, then

\[
f(\cos \theta) = \sum_{i=0}^{\infty} a_i T_i(\cos \theta) = \sum_{i=0}^{\infty} a_i \cos(i\theta), \quad 0 \leq \theta \leq \pi,
\]

which is just the Fourier cosine series for the function $f(\cos \theta)$.

Hence, it is a series we could evaluate (using numerical integration if necessary).

Now, suppose the series converges rapidly so that,

\[
|a_{n+1}| \gg |a_{n+2}| \gg |a_{n+3}| \gg \ldots
\]

so a few terms are a good approximation of the function.
Let $\Psi(x) = \sum_{i=0}^{n} a_i T_i(x)$ then
\[
f(x) - \Psi(x) = a_{n+1} T_{n+1}(x) + a_{n+2} T_{n+2}(x) + \ldots
\]
\[
\simeq a_{n+1} T_{n+1}(x),
\]
or, the error is dominated by the leading term $a_{n+1} T_{n+1}(x)$. Now $T_{n+1}(x)$ equi-oscillates $(n+2)$ times on $[-1,1]$.

If $f(x) - \Psi(x) = a_{n+1} T_{n+1}(x)$, then $\Psi(x)$ would be the minimax polynomial of degree $n$ to $f(x)$. Since
\[
f(x) - \Psi(x) \simeq a_{n+1} T_{n+1}(x),
\]
$\Psi(x)$ is not the minimax but is a polynomial that is ‘close’ to the minimax, as long as $a_{n+2}, a_{n+3}, \ldots$ are small compared to $a_{n+1}$.

The actual error almost equi-oscillates on $(n+2)$ points.

**Example 4.6.1:** Find the minimax quadratic approximation to $f(x) = (1 - x^2)^{1/2}$ in the interval $[-1,1]$.

First, we note that if $x = \cos \theta$ then $f(\cos \theta) = (1 - \cos^2 \theta)^{1/2} = \sin \theta$ and the interval $x \in [-1,1]$ becomes $\theta \in [0,\pi]$.

The Fourier cosine series for $\sin \theta$ on $[0,\pi]$ is given by
\[
\sin \theta = \frac{2}{\pi} - \frac{4}{\pi} \left[ \frac{\cos 2\theta}{3} + \frac{\cos 4\theta}{15} + \frac{\cos 6\theta}{35} + \ldots \right]
\]

So with $x = \cos \theta$, we have
\[
(1 - x^2)^{1/2} = \frac{2}{\pi} - \frac{4}{\pi} \left[ \frac{T_2(x)}{3} + \frac{T_4(x)}{15} + \frac{T_6(x)}{35} + \ldots \right],
\]
where $-1 \leq x \leq 1$.

Thus let use consider the quadratic
\[
p_2(x) = \frac{2}{\pi} - \frac{4}{\pi} \frac{T_2(x)}{3} = \frac{2}{\pi} - \frac{4}{3\pi} (2x^2 - 1)
\]
\[
= \frac{2}{3\pi} (3 - 2(2x^2 - 1)) = \frac{2}{3\pi} (5 - 4x^2).
\]

The error
\[
f(x) - p_2(x) \approx \frac{4}{\pi} \frac{T_4(x)}{15},
\]
which oscillates $4 + 1 = 5$ times in $[-1,1]$. At least 4 equi-oscillation points are required for $p_2(x)$ to be the minimax approximation of $(1 - x^2)^{1/2}$, so we need to see whether the above oscillation points are of equal amplitude.

$T_4(x)$ has extreme values when $8x^4 - 8x^2 + 1 = \pm 1$, i.e. at

$$ x = 0, \ x = 1, \ x = -1, \ x = 1/\sqrt{2} \text{ and } x = -1/\sqrt{2}. $$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$(1 - x^2)^{1/2}$</th>
<th>$p_2(x)$</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td>1</td>
<td>10/3\pi</td>
<td>-0.0610</td>
</tr>
<tr>
<td>$x = \pm 1/\sqrt{2}$</td>
<td>1/\sqrt{2}</td>
<td>2/\pi</td>
<td>0.0705</td>
</tr>
<tr>
<td>$x = \pm 1$</td>
<td>0</td>
<td>2/3\pi</td>
<td>-0.2122</td>
</tr>
</tbody>
</table>

So the error oscillates but not equally. Hence, $p_2(x)$ is not quite the minimax approximation to $f(x) = (1 - x^2)^{1/2}$, but it is a good first approximation.

The true minimax quadratic to $(1 - x^2)^{1/2}$ is actually $(\frac{9}{8} - x^2) = (1.125 - x^2)$, and thus our estimate of $(1.061 - 0.8488x^2)$ is not bad.

### 4.7 Economisation of a Power Series

Another way of exploiting the properties of Chebyshev polynomials is possible for functions $f(x)$ for which a power series exists.

Consider the function $f(x)$ which equals the power series

$$ f(x) = \sum_{n=1}^{\infty} a_n x^n. $$

Let us assume that we are interested in approximating $f(x)$ with a polynomial of degree $m$.

One such approximation is

$$ f(x) = \sum_{n=1}^{m} a_n x^n + R_m, $$

which has error $R_m$. Can we get a better approximation of degree $m$ than this?

Yes! A better approximation may be found by finding a function $p_m(x)$ such that $f(x) - p_m(x)$ equi-oscillates at least $m + 2$ times in the given interval.

Consider the truncated series of degree $m + 1$

$$ f(x) = \sum_{n=1}^{m} a_n x^n + a_{m+1} x^{m+1} + R_{m+1}. $$

The Chebyshev polynomial of degree $m + 1$, equi-oscillates $m + 2$ times, and equals

$$ T_{m+1}(x) = 2^m x^{m+1} + t_{m-1}(x), $$
where $t_{m-1}$ are the terms in the Chebyshev polynomial of degree at most $m - 1$. Hence, we can write

$$x^{m+1} = \frac{1}{2^m} (T_{m+1}(x) - t_{m-1}(x)).$$

Substituting for $x^{m+1}$ in our expression for $f(x)$ we get

$$f(x) = \sum_{n=1}^{m} a_n x^n + \frac{a_{m+1}}{2^m} (T_{m+1}(x) - t_{m-1}(x)) + R_{m+1}.$$

Re-arranging we find a polynomial of degree at most $m$,

$$p_m(x) = \sum_{n=1}^{m} a_n x^n - \frac{a_{m+1}}{2^m} t_{m-1}(x).$$

This polynomial will be a pretty good approximation to $f(x)$ since

$$f(x) - p_m(x) = \frac{a_{m+1}}{2^m} T_{m+1}(x) + R_{m+1},$$

which oscillates $m + 2$ times almost equally provided $R_{m+1}$ is small.

Although $p_m(x)$ is not the minimax approximation to $f(x)$ it is close and the error

$$\frac{a_{m+1}}{2^m} T_{m+1}(x) + R_{m+1} \leq \frac{a_{m+1}}{2^m} + R_{m+1},$$

since $|T_{m+1}(x)| \leq 1$, is generally a lot less than the error $R_m$ for the truncated power series of degree $m$.

This process is called the Economisation of a power series.

**Example 4.7.1:** The Taylor expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + R_7,$$

where

$$R_7 = \frac{x^7}{7!} \frac{d^7}{dx^7} (\sin x)_{x=\theta} = \frac{x^7}{7!} (-\cos \theta).$$

For $x \in [-1, 1]$, $|R_7| \leq \frac{1}{7!} \approx 0.0002$.

However,

$$\sin x = x - \frac{x^3}{3!} + R_5,$$

where

$$R_5 = \frac{x^5}{5!} \frac{d^5}{dx^5} (\sin x)_{x=\theta} = \frac{x^5}{5!} (\cos \theta),$$

so $|R_5| \leq \frac{1}{5!} \approx 0.0083$. The extra term makes a big difference!

Now suppose we express $x^5$ in terms of Chebyshev polynomials,

$$T_5(x) = 16x^5 - 20x^3 + 5x,$$
so

\[ x^5 = \frac{T_5(x) + 20x^3 - 5x}{16}. \]

Then

\[
\sin x = x - \frac{x^3}{6} + \frac{1}{5!} \left( \frac{T_5(x) + 20x^3 - 5x}{16} \right) + R_7
\]

\[ = x \left( 1 - \frac{1}{16 \times 4!} \right) - \frac{x^3}{6} \left( 1 - \frac{1}{16} \right) + \frac{1}{16 \times 5!} T_5(x) + R_7. \]

Now \(|T_5(x)| \leq 1\) for \(x \in [-1, 1]\) so if we ignore the term in \(T_5(x)\) we obtain

\[
\sin x = x \left( 1 - \frac{1}{16 \times 4!} \right) - \frac{x^3}{6} \times \frac{15}{16} + \text{Error}
\]

where

\[
|\text{Error}| \leq |R_7| + \frac{1}{16 \times 5!} |T_5(x)|,
\]

\[ \leq 0.0002 + \frac{1}{16 \times 120} = 0.0002 + \frac{1}{1920} \]

\[ \leq 0.0002 + 0.00052 \simeq 0.0007. \]

This new cubic has maximum error of about 0.0007, compared with 0.0083 for \(x - \frac{x^3}{6}\).