Slow magnetohydrodynamic waves
in the solar atmosphere

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There is increasingly strong observational evidence that slow magnetoacoustic modes arise in the solar atmosphere, either as propagating or standing waves. Sunspots, coronal plumes and coronal loops all appear to support slow modes. Here we examine theoretically how the slow mode may be extracted from the magnetohydrodynamic equations, considering the special case of a vertical magnetic field in a stratified medium: the slow mode is described by the Klein–Gordon equation. We consider its application to recent observations of slow waves in coronal loops.

Keywords: Sun; coronal oscillations; slow magnetohydrodynamic waves; Klein–Gordon equation

1. Introduction

There is considerable observational evidence for the occurrence of slow magnetoacoustic waves in the solar atmosphere. Slow waves are believed to arise: in sunspots, in the form of 3 min oscillations (Bogdan 2000); in kilogauss thin flux tubes (Roberts & Ulmschneider 1997) and network regions (McAteer et al. 2002, 2003; Hasan et al. 2003); in the footpoints of coronal loops (Berghmans & Clette 1999; Nightingale et al. 1999; De Moortel et al. 2000, 2002a–d; Robbrecht et al. 2001; Tsiklauri & Nakariakov 2001; King et al. 2003), where both 3 and 5 min periodicities have been detected; in intensity oscillations in the upper transition zone moss (De Pontieu et al. 2003); in coronal plumes (Ofman & Wang 2002); in coronal loops generally (Sakurai et al. 2002); and as global oscillations of hot coronal loops (Kliem et al. 2002; Wang et al. 2002, 2003a,b). Both standing and propagating slow waves may occur: standing waves are found in the hot loops (Ofman & Wang 2002), and propagating waves are seen in sunspots, moss, plumes and in loop footpoint regions. The slow mode observations are generally quite distinct from the other oscillations found in coronal loops, such as the global kink mode detected in Transition Region and Coronal Explorer (TRACE) observations (see the reviews by Aschwanden 2003, 2004), the fast waves found by Solar Eclipse Coronal Imaging System (SECIS) (Williams et al. 2001, 2002), or the quasi-periodic pulsations found in radio wavelengths and ascribed to global sausage modes (Nakariakov et al. 2003). All
these waves, fast or slow, are potentially valuable sources of seismic information, providing a coronal seismology (Roberts et al. 1984; Roberts 1986; Nakariakov et al. 1999; Nakariakov & Ofman 2001); see the reviews by Roberts (2000, 2004) and Nakariakov (2004).

The theoretical description of such modes is complicated, not least by the geometry of solar magnetic structures, and such developments are vital if coronal seismology is to progress. One approach has been to consider pure acoustic waves, ignoring the role of the magnetic field except to admit that it renders sound wave propagation as one-dimensional. By considering here a specific situation, namely that of a stratified medium with a vertical magnetic field, we show that such an approach is broadly valid as a description of the slow magnetoacoustic wave, and we calculate the changes that the magnetic field introduces. We discuss how the slow mode may be extracted from the magnetohydrodynamic (MHD) equations. Our analysis leads to the Klein–Gordon equation, similar to that found in vertically propagating sound waves (Lamb 1932) or to MHD waves in isolated thin magnetic flux tubes (Roberts 1981b; Rae & Roberts 1982; Spruit & Roberts 1983); the Klein–Gordon equation has recently been used as a means of explaining solar spicules (De Pontieu et al. 2004). A recent review of the role of the Klein–Gordon equation in magnetic flux tubes is provided in Roberts (2004). Finally, we illustrate our analysis by applying it to the case of oscillating coronal loops, considering the standing waves detected in hot loops by the spacecraft Solar and Heliospheric Observatory/Solar Ultraviolet Measurements of Emitted Radiation (SUMER) (Wang 2004) and the propagating waves in loop footpoints detected by TRACE (De Moortel et al. 2000, 2002a–d).

What is the slow MHD mode? We know from studies of MHD waves in a uniform medium (e.g. Roberts 1985) that the slow mode can be viewed as a mixture of an Alfvén wave and a sound wave. The Alfvénic branch provides the slow mode with the property of being field-guided: in a uniform medium, the slow mode group velocity never departs by more than $\tan^{-1}(1/2) (=26.6^\circ)$ from the direction of the applied field. The sound wave branch provides the mode’s compressibility, by which pressure variations are out of phase with magnetic pressure variations. In a strong field, the flows in the slow wave are predominantly along the applied magnetic field. Moreover, in magnetic flux tubes, the slow mode is even more strongly wave guided along the tube (e.g. Roberts & Webb 1978; Edwin & Roberts 1983). Our question here, then, is: how can we extract information about slow modes from the MHD equations without the need to calculate the behaviour of all the MHD modes?

2. Waves in uniform media

To motivate our approach consider first the familiar problem of magnetoacoustic waves in a uniform, unstratified medium. The divergence $\Delta = \text{div } \mathbf{v}$ of the velocity field $\mathbf{v}$ satisfies the wave equation (Lighthill 1960; Cowling 1976; Roberts 1985)

$$\frac{\partial^4 \Delta}{\partial t^4} - \left( c_S^2 + c_A^2 \right) \frac{\partial^2}{\partial t^2} \nabla^2 \Delta + c_S^2 c_A^2 \frac{\partial^2}{\partial z^2} \nabla^2 \Delta = 0,$$

where $c_S = (\gamma p_0/\rho_0)^{1/2}$ and $c_A = (B_0^2/\mu_0 \rho_0)^{1/2}$ are the sound and Alfvén speeds in a magnetic atmosphere with plasma pressure $p_0$ and density $\rho_0$. The applied
magnetic field $B_0 \hat{z}$ is aligned with the $z$-axis and $\nabla^2$ denotes the three-dimensional Laplacian operator.

Equation (2.1) describes both slow and fast magnetoacoustic waves. In terms of a Fourier analysis,

$$\Delta = \Delta_0 \exp i(\omega t - k \cdot \mathbf{r})$$

for modes of constant amplitude $\Delta_0$, frequency $\omega$ and wave vector $\mathbf{k} = (k_x, k_y, k_z)$ with magnitude $k$, equation (2.1) yields the familiar dispersion relation

$$\omega^4 - k^2 (c_S^2 + c_A^2) \omega^2 + k_x^2 k_y^2 c_S^2 c_A^2 = 0,$$

which may be rearranged into the form

$$k_x^2 + k_y^2 = n_0^2,$$

where we have introduced the effective transverse wavenumber $n_0$ and the tube speed $c_t$ through (Roberts 1981a)

$$n_0^2 = \frac{(\omega^2 - k_z^2 c_S^2)(\omega^2 - k_z^2 c_A^2)}{(c_S^2 + c_A^2)(\omega^2 - k_z^2 c_t^2)}, \quad c_t^2 = \frac{c_S^2 c_A^2}{c_S^2 + c_A^2}.$$  \hspace{1cm} (2.4)

The fast and slow magnetoacoustic waves are the two solutions (in $\omega^2$) of the dispersion relation (2.2). From the form (2.3), we see that the slow wave is such that $k_x^2 c_t^2 \leq \omega^2 \leq \min(k_x^2 c_S^2, k_x^2 c_A^2)$ and the fast wave has $\omega^2 \geq \max(k_x^2 c_S^2, k_x^2 c_A^2)$. In the limit of $4k_x^2 c_t^2 \ll k^2 (c_S^2 + c_A^2)$, we see that the slow mode gives

$$\omega^2 \approx k_x^2 c_t^2 \left[ 1 + \left( \frac{k_z^2}{k_x^2} \right) \frac{c_t^2}{c_S^2 + c_A^2} \right].$$  \hspace{1cm} (2.5)

Thus, the slow wave propagating at an angle that is almost orthogonal to the applied magnetic field ($k_x^2 \ll k^2$) or propagating in a strong ($c_A^2 \gg c_S^2$) or weak ($c_A^2 \ll c_S^2$) magnetic field (so that $c_t^2 \ll c_S^2 + c_A^2$) does so with a phase speed that is $c_t$ along the field. Alternatively, if we consider a mode that is sinusoidal across the magnetic field but propagating along the field, then for slow modes that are rapidly oscillatory across the field—having a very short transverse length—the longitudinal phase speed is close to $c_t$.

How can we extract this information from equation (2.1)? Consider, for simplicity, the two-dimensional case for which the motions are $\mathbf{v} = (v_x, 0, v_z)$ with $\partial/\partial y = 0$. Introduce the spatial and temporal scalings

$$x = \epsilon X, \quad z = Z, \quad t = T,$$  \hspace{1cm} (2.6)

so that $\partial/\partial x = (1/\epsilon)(\partial/\partial X)$, whereas $\partial/\partial z = \partial/\partial Z$ and $\partial/\partial t = \partial/\partial T$. Then equation (2.1) becomes

$$\epsilon^2 \frac{\partial^2 \Delta}{\partial T^2} - (c_S^2 + c_A^2) \frac{\partial^2}{\partial T^2} \left( \frac{\partial^2 \Delta}{\partial X^2} + \epsilon^2 \frac{\partial^2 \Delta}{\partial Z^2} \right) + c_S^2 c_A^2 \frac{\partial^2}{\partial Z^2} \left( \frac{\partial^2 \Delta}{\partial X^2} + \epsilon^2 \frac{\partial^2 \Delta}{\partial Z^2} \right) = 0.$$  \hspace{1cm} (2.7)

For $\epsilon \ll 1$, the transverse derivatives are much larger than the longitudinal ones ($X$ and $Z$ being assumed of comparable order), so that to lowest order in $\epsilon$ this yields

$$\frac{\partial^2 \phi}{\partial T^2} = c_t^2 \frac{\partial^2 \phi}{\partial Z^2}; \quad \phi = \frac{\partial^2 \Delta}{\partial X^2}.$$  \hspace{1cm} (2.8)

We have thus obtained the one-dimensional wave equation, giving $c_t$ as the longitudinal speed of propagation.

It is of interest to examine the perturbations. The velocity components \(v_x\) and \(v_z\) are in general related by (e.g. Roberts 1981a)

\[
v_z = \frac{ik_z c_S^2}{(\omega^2 - k_z^2 c_S^2)} \frac{\partial v_x}{\partial x}.
\]

So under the scaling (2.6) we see that \(v_x\) is comparable with \(\epsilon v_z\), since

\[
v_z = -\frac{k_z}{k_z} \left( \frac{c_S^2 + c_A^2}{c_S^2} \right) v_x.
\]

Turning to the plasma pressure perturbation \(p\) and the magnetic pressure perturbation \(p_m\) (= \((1/\mu)B_0 \cdot B\), for perturbed field \(B\)), we have (Roberts 1981a)

\[
p = \frac{\omega}{k_z} \rho_0 v_z, \quad p_m = \frac{c_A^2}{c_S^2} \left( \frac{\omega^2 - k_z^2 c_S^2}{k_z c^2} \right) \rho_0 v_z,
\]

and so \(p\) and \(p_m\) are comparable in order to \(v_z\). However, the perturbation \(p_T = p + p_m\) in total pressure is of lower order than \(v_z\), since

\[
p_T = \rho_0 \frac{c_1}{k_z} \frac{k_z^2 c_S^2}{c_S^2} (\omega^2 - k_z^2 c_t^2) v_z.
\]

Thus,

\[
p_T = \rho_0 c_1 v_z \frac{k_z}{k_z} \frac{c_S^2}{c_S^2},
\]

and so \(p_T/\rho_0 c_1 v_z\) is of order \(k_z^2 c_t^2/k^2 c_S^2\), i.e. of order \(\epsilon^2\).

Much of these ideas may be extended to structured media, including the standard model of a coronal loop. This consists of a uniform plasma in a cylindrical tube surrounded by a uniform environment (e.g. Edwin & Roberts 1983). Then solutions of the wave equation (2.1) in cylindrical coordinates arise; inside a flux tube of radius \(r=a\), the amplitude of the total pressure perturbation (describing both fast and slow magnetoacoustic waves) is of the form

\[
p_T = AJ_n(n_0 r), \quad r < a,
\]

where \(J_n\) denotes the Bessel function of order \(n\) and \(n_0\) is defined in (2.4). In such magnetic flux tubes, the slow mode arises with speed \(\omega/k_z\) close to \(c_t\) (Edwin & Roberts 1983). Thus, \(n_0^2\) tends to be large in slow waves and the transverse derivative \(\partial p_T/\partial r\) is large. Consequently, the analysis presented above applies also to cylindrical coordinates, with only minor changes.

### 3. Modes in a stratified medium

Suppose now that the plasma is stratified under gravity, \(-g\hat{z}\), and embedded in a uniform, vertical, magnetic field \(\hat{B}_0 \cdot \hat{z}\). Then the equilibrium pressure \(p_0(z)\) and density \(\rho_0(z)\) are related by the requirement of hydrostatic pressure balance

\[
p_0'(z) = -g\rho_0(z), \quad (3.1)
\]

the dash (’) denoting differentiation (of an equilibrium quantity) with respect to the height \(z\). Combined with the ideal gas law, equation (3.1) yields

\[
p_0(z) = p_0(0)e^{-\mu(z)}, \quad \rho_0(z) = p_0(0)\frac{A(0)}{A(z)} e^{-\mu(z)},
\]

**Phil. Trans. R. Soc. A (2006)**
giving the equilibrium pressure and density distributions (relative to an arbitrary reference level $z=0$) in terms of the pressure scale height $A(z) = \frac{p_0(z)}{g \rho_0(z)}$ and the integral

$$n(z) = \int_0^z dz'/A(z').$$

Linear perturbations about the equilibrium (3.1) and (3.2) are readily determined. Two-dimensional perturbations $v = (v_x, 0, v_z)$ satisfy the pair of wave equations (Ferraro & Plumpton 1958)

$$\left( c_A^2 \frac{\partial^2}{\partial z^2} + (c_S^2 + c_A^2) \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) v_x + \left( c_S^2 \frac{\partial}{\partial z} - g \right) \frac{\partial v_z}{\partial \tau} = 0,$$

$$\left( c_S^2 \frac{\partial}{\partial z} - (\gamma - 1) g \right) \frac{\partial v_x}{\partial \tau} + \left( c_S^2 \frac{\partial^2}{\partial z^2} - \gamma g \frac{\partial}{\partial z} - \frac{\partial^2}{\partial t^2} \right) v_z = 0,$$

where now the sound speed $c_S(z)$ and the Alfvén speed $c_A(z)$ are in general functions of height $z$. When $g=0$ these equations imply the fourth-order wave equation (2.1).

Equations (3.3) and (3.4) govern the behaviour of fast and slow magnetoacoustic waves in a stratified plasma, including the coupling to gravity modes. In the absence of a magnetic field they describe the $p$- and $g$-modes of helioseismology. In the presence of a magnetic field, equations (3.3) and (3.4) may be used to describe compressible convection in a vertical magnetic field, though our interest here is in the waves the system supports. In fact, for an isothermal atmosphere, equations (3.3) and (3.4) may be Fourier analysed in $x$ and $t$ to produce a fourth-order ordinary differential equation for the vertical spatial dependence, and indeed this equation may be solved in terms of Meijer functions (Zhugzhda 1979; Webb 1980; Zhugzhda & Dzhalilov 1982). However, the information obtained this way can remain obscure, though Cally (2001) has shown that the solutions can be written more informatively in terms of hypergeometric functions $\, _2F_3$. Nonetheless, it remains desirable to extract out a simple description of the slow mode.

We are interested, then, in the behaviour of slow magnetoacoustic modes. In a strong magnetic field, slow modes are characterized by motions that are predominantly along the field (e.g. Cowling 1976). Motivated by the scalings introduced earlier in a uniform medium, we write

$$x = \epsilon X, \quad z = Z, \quad t = T, \quad v_x = \epsilon u_x, \quad v_z = u_z.$$  \hspace{1cm} (3.5)

Thus, for small $\epsilon$, motions are predominantly along the magnetic field and of short transverse wavelength. It is interesting to note that such scalings have been used in the analysis of slow modes within resonant layers in a magnetically structured plasma (Ruderman et al. 1997; Ballai et al. 1998; Ruderman 2000), but, as far as we are aware, our application here is its first use in a stratified medium.

Under the scaling (3.5) the form of equation (3.4) is unchanged, but equation (3.3) now reads

$$(c_S^2 + c_A^2) \frac{\partial^2 u_x}{\partial X^2} + \left( c_S^2 \frac{\partial}{\partial Z} - g \right) \frac{\partial u_z}{\partial X} = \epsilon^2 \left( \frac{\partial^2}{\partial T^2} - c_A^2 \frac{\partial^2}{\partial Z^2} \right) u_x.$$  \hspace{1cm} (3.6)
Thus, for $\epsilon^2 \ll 1$ we have

$$
(c_s^2 + c_A^2) \frac{\partial^2 u_x}{\partial X^2} + \left( c_s^2 \frac{\partial}{\partial Z} - g \right) \frac{\partial u_z}{\partial X} = 0,
$$

(3.7)

which can be integrated immediately with respect to $X$ to relate $\partial u_z/\partial X$ to $u_x$.

Returning to the original (unscaled) variables, we thus see that equation (3.3) is replaced by

$$
(c_s^2 + c_A^2) \frac{\partial v_x}{\partial x} + \left( c_s^2 \frac{\partial}{\partial z} - g \right) v_z = 0,
$$

(3.8)

with equation (3.4) unchanged. The scaling (3.5) has eliminated the fast magnetoacoustic modes (and, therefore, $p$-modes also) but retained the behaviour of the slow mode (and its unstable counterpart, convection).

Elimination of $\partial v_x/\partial x$ between equations (3.4) and (3.8) yields

$$
\frac{\partial^2 v_z}{\partial t^2} - c_t^2 \frac{\partial^2 v_z}{\partial Z^2} + \gamma g \left( \frac{c_t}{c_s} \right)^4 \frac{\partial v_z}{\partial Z} + \frac{c_t^2}{c_A^2} \left( \omega_g^2 + \frac{g}{A} \frac{c_t^2}{c_s^2} \right) v_z = 0,
$$

(3.9)

where $\omega_g^2$ is the square of the buoyancy (Brunt–Väisälä) frequency,

$$
\omega_g^2 = -g \left( \frac{g}{c_s^2} + \frac{\rho_0}{\rho_0} \right).
$$

Equation (3.9) generalizes (2.8) to the case of a vertical magnetic field in a stratified medium.

It is convenient to introduce

$$
Q(z, t) = \left( \frac{\rho_0(z) c_t^2(z)}{\rho_0(0) c_t^2(0)} \right)^{1/2} v_z(z, t),
$$

(3.10)

which eliminates the first derivative term involving $\partial v_z/\partial Z$. Then, equation (3.9) is cast in the form of the Klein–Gordon equation, viz.

$$
\frac{\partial^2 Q}{\partial t^2} - c_t^2(z) \frac{\partial^2 Q}{\partial Z^2} + \Omega^2(z) Q = 0,
$$

(3.11)

where

$$
\Omega^2 = c_t^2 \left\{ \frac{1}{4 A^2} \left( \frac{c_t}{c_s} \right)^4 - \frac{1}{2} \gamma g \left( \frac{c_t^2}{c_s^2} \right)' + \frac{1}{c_A^2} \left( \omega_g^2 + \frac{g}{A} \frac{c_t^2}{c_s^2} \right) \right\}.
$$

(3.12)

Equation (3.11) is the basic equation of our discussion. The Klein–Gordon equation arises also in the description of vertically propagating sound waves and in MHD waves in a thin magnetic flux tube (Rae & Roberts 1982; Spruit & Roberts 1983; Sutmann et al. 1998; Hasan et al. 2003; Musielak & Ulmschneider 2003a,b; Noble et al. 2003); see Roberts (2004) for a recent review. Indeed, equation (3.11) may be used also to discuss stability to convection in a uniform vertical magnetic field, yielding the Gough–Tayler criterion (Gough & Tayler 1966; see also Webb & Roberts 1978; Webb 1980). However, our interest here is on wave propagation and we see that slow waves in a vertical magnetic field behave, under the scaling (3.5), as waves in a straight flux tube, and evidently possess properties akin to those of an isolated tube (see the review by Roberts & Ulmschneider 1997).

To elucidate those properties first notice the two extremes by which the speed $c_t(z)$ and the frequency $\Omega(z)$ may change. In a very strong magnetic field (or cold

Phil. Trans. R. Soc. A (2006)
plasma), with \(c_S/c_A \ll 1\), corresponding most closely to (say) the upper atmosphere of a sunspot or in a coronal loop, we have

\[
\frac{c_t}{c_s} = \frac{\Omega^2}{4A^2}(1 + 2A').
\]

(3.13)

We here recognize a description of sound waves constrained to propagate one-dimensionally along rigid magnetic field lines. The frequency \(\Omega(z)\) has become the acoustic cutoff frequency of a stratified atmosphere (Lamb 1932). In the opposite extreme of a weak field \((c_A \ll c_S)\), such as arises in the sub-photospheric layers of a sunspot, we have

\[
\frac{c_t}{c_A} = \frac{\Omega^2}{\omega_g^2},
\]

(3.14)

giving the Alfvén speed as the speed of propagation and the buoyancy frequency \(\omega_g\) as the frequency \(\Omega\).

Equation (3.11) has a simple interpretation when \(c_t\) and \(\Omega\) are constants. If disturbances in \(Q\) are generated impulsively, then a wave propagates with wavefront speed \(c_t\) and trails a wake oscillating with the frequency \(\Omega\) (Lamb 1909, 1932; Rae & Roberts 1982). If, however, disturbances are driven by a sustained oscillation with fixed frequency \(\omega = \omega_0\) (corresponding, say, to an oscillating base at \(z=0\)), then propagation occurs only for \(\omega_0 > \Omega\). The frequency \(\Omega\) is, thus, a cutoff frequency.

It is convenient to Fourier analyse in time by writing

\[
v_z(z, t) = v_z(z)e^{i\omega t}, \quad Q(z, t) = Q(z)e^{i\omega t}
\]

for frequency \(\omega\). Then equation (3.11) becomes

\[
\frac{d^2Q}{dz^2} + \frac{\omega^2 - \Omega^2(z)}{c_t^2(z)}Q = 0.
\]

(3.15)

Equations of this form have been discussed in Syrovatskii & Zugzda (1968), Webb (1980) and Moreno-Insertis & Spruit (1989), though in these works the application is to photospheric problems.

4. Slow modes in coronal loops

Many recent treatments of coronal oscillations have approached the modelling by considering analytically or through numerical simulations the one-dimensional acoustic wave. From the viewpoint of our treatment, it is apparent from the form of equation (3.15) that such approaches are for many purposes quite adequate; in effect, such treatments (e.g. Nakariakov et al. 2000, 2004; Tsiklauri & Nakariakov 2001; James 2003; Mendoza-Briceno et al. 2004; Tsiklauri et al. 2004) effectively replace \(\Omega\) by its acoustic value and \(c_t\) by the sound speed \(c_s\).

The strong magnetic field of a coronal loop acts as a waveguide for sound waves, which then propagate one-dimensionally as the slow magnetoacoustic wave.

Consider the specific application of our analysis to a coronal loop. To apply our discussion to a coronal loop, we need to employ boundary conditions that allow in part for the curved geometry of a loop; we can do this by treating only that section of a loop that extends from the base to its apex. This circumvents some of the problems associated with loop geometry. An alternative approach is to allow variations in the magnitude of the gravitational acceleration, so that the
effective value of $g$ is allowed to vary along the length of a loop, reflecting the changing angle that the central axis of the loop makes with the vertical; this approach has been successfully employed in acoustic calculations by Nakariakov et al. (2000) and Tsiklauri & Nakariakov (2001). Indeed, allowing a variable $g$ in the basic equations we have discussed leads once more to equation (3.9) but with the addition on the left-hand side of a term $(c_t^2/c_S^2)g'v_z$. Consequently, the cutoff frequency $\Omega$ is modified. However, a discussion of this effect would take us too far afield and so we leave the topic for a future investigation. Accordingly, we retain our assumption of constant $g$. Moreover, we assume the loop to be isothermal; with the exception of the footpoint regions, the temperature variation along a coronal loop is not pronounced, so the assumption of isothermality (constant sound speed $c_S$ and pressure scale height $A$) is appropriate. In an isothermal medium with constant $g$, the Alfvén speed increases with height exponentially fast, on a scale of twice the pressure scale height

$$c_A(z) = c_A(0)\exp\left(\frac{z}{2A}\right).$$

So, for example, in a loop of length 200 Mm, with pressure scale height $A=5\times10^4$ km, the Alfvén speed increases by a factor of $e$ ($\approx 2.7$) from the footpoint to the apex of the loop.

Consider the cutoff frequency $\Omega$ in an isothermal loop with strong magnetic field, so that $c_A >> c_S$; then equation (3.12) yields (for $\gamma \neq 1$)

$$\frac{\Omega^2}{c_t^2} \approx \frac{1}{A^2} \left[ \frac{1}{4} - \left( 1 - \frac{1}{\gamma} \right)^2 \frac{c_S^2}{c_A^2} \frac{v_z^2}{c_t^2} \right].$$ (4.1)

Thus, magnetism decreases the ratio of $\Omega^2/c_t^2$, and it decreases the propagation speed $c_t$; thus, the cutoff frequency $\Omega$ falls in the presence of a strong magnetic field

$$\Omega^2 \approx \frac{c^2_S}{4A^2} \left[ 1 - \left( 1 + 4 \left( 1 - \frac{1}{\gamma} \right)^2 \right) \frac{c_S^2}{c_A^2} \right], \quad (c_S^2 \ll c_A^2).$$ (4.2)

With $\gamma = 5/3$ we obtain

$$\Omega \approx \frac{c_S}{2A} \left[ 1 - \frac{41}{50} \left( \frac{c_S^2}{c_A^2} \right) \right], \quad (c_S^2 \ll c_A^2).$$ (4.3)

The effect is relatively small; for a sound speed of $c_S = 150$ km s$^{-1}$ and an Alfvén speed of $c_A = 10^3$ km s$^{-1}$, it amounts to a reduction in $\Omega$ of 1.8% below its acoustic value, $\Omega_a = c_S/(2A)$. A similar reduction occurs in the propagation speed $c_t$, which falls below the sound speed $c_S$ by about 1.1%. With $c_S = 150$ km s$^{-1}$, gravitational constant $g = 0.27$ km s$^{-2}$ and $\gamma = 5/3$, we obtain a pressure scale height of $A = 5\times10^4$ km (or 50 Mm) and an acoustic cutoff frequency of $\Omega_a = 1.5\times10^{-3}$ s$^{-1}$ (with corresponding period $2\pi/\Omega = 4189$ s or almost 70 min).

Turning now to the observational aspects, we note that both standing and propagating slow waves have been observed in loops. We discuss them separately.

(a) Standing waves

To apply our analysis to the observations of standing slow waves in coronal loops (Öfman & Wang 2002; Wang et al. 2002, 2003a, b), we suppose that $Q = 0$ at
the footpoint of a loop \((z=0)\) and that \(Q\) has a maximum \((dQ/dz=0)\) at the loop apex \((z=L/2)\), the total loop length being \(L\). Both these boundary conditions may be modified to treat the complexities of loop footpoints (James 2003) and loop curvature, but for our purposes here a simpler treatment is more illuminating. With these two boundary conditions, equation (3.15) has solution (taking \(c_t\) and \(U\) as approximately constant)

\[
Q = Q_0 \sin \left[ \left( \frac{\omega^2 - \Omega^2}{c_t^2} \right) \frac{1}{2} \right],
\]

(4.4)

with

\[
\omega^2 = \Omega^2 + \left( \frac{2\pi c_t}{L} \right)^2 \left( n - \frac{1}{2} \right)^2,
\]

(4.5)

where \(n\) is a positive integer. Thus, the principal mode \((n=1)\) has a period \(P(=2\pi/\omega)\) given by

\[
P = \frac{2L}{c_t} \frac{1}{[1 + (L\Omega/c_t)^2]^{1/2}} = \frac{2L}{c_t} \frac{1}{[1 + (L^2/L_c^2)]^{1/2}},
\]

(4.6)

where \(L_c = \pi c_t/\Omega\) denotes the effective loop length introduced by stratification. The period of the standing wave is, thus, reduced by the effect of stratification; the square root factor in (4.6) indicates by how much the period \(P=2L/c_t\) obtained in the \(g=0\) case (Edwin & Roberts 1983; Roberts et al. 1984) is modified as a consequence of stratification. Evidently, the importance (or otherwise) of stratification is determined by the magnitude of \((L/L_c)^2\); if \(L \ll L_c\), then stratification is unimportant (as regards the period of a slow wave), but if this term is of order unity or larger then stratification has a pronounced effect. Since \(c_t \sim c_S\) and \(\Omega \sim \Omega_a\), the acoustic values, we have \(L_c \sim 2\pi A\) and the importance of gravity is determined by the magnitude of \((L/2\pi A)^2\). The effect of gravity on short \((L \ll L_c)\) loops is negligible, their period being determined principally by the travel time \(2L/c_t\) in an unstratified loop.

Many of the hot loops observed by the SUMER instrument (Wang et al. 2003a,b) are short in the above sense. A sound speed of \(c_S = 370\) km s\(^{-1}\) (corresponding to the high temperature of 6.3 MK) leads to a pressure scale height of \(A = c_S^2/\gamma g = 3 \times 10^5\) km; thus, \(L_c = 1900\) Mm, far larger than the observed typical loop length of \(L = 191\) Mm. Hence, gravitational effects on the period \(P\) of slow modes in hot SUMER loops are negligible, and the period is determined essentially by the travel time \(2L/c_t\). An observed period of \(P = 17.6\) min thus implies a tube speed of \(c_t = 362\) km s\(^{-1}\) and this in turn implies an Alfvén speed of \(c_A = 1750\) km s\(^{-1}\). On the other hand, loops observed by TRACE are much cooler than the hot SUMER loops and have sound speeds of order \(c_S = 150\) km s\(^{-1}\); this produces \(L_c = 300\) Mm, which is comparable with the lengths of many TRACE loops (which have an average loop length of 220 Mm). However, slow mode standing oscillations of TRACE loops have yet to be observed.

(b) Propagating waves

Consider the possibility of slow waves propagating in a loop. If insufficient time has elapsed for the wave to reach the far end of a loop, or if decay mechanisms act
efficiently to prevent a slow wave surviving to the far ends of a loop, then the wave will propagate freely. The observations by De Moortel et al. (2002a–d) would seem to be in this category, since the waves appear to penetrate only about 10% of a loop from the footpoint before vanishing. We can envisage two aspects: that the waves are impulsively excited, or that the waves are driven by an oscillation in the lower atmosphere. Impulsively excited waves are likely to arise when an energetic event, such as a flare, occurs; driven waves are likely to arise from p-modes (5 min period) or from chromospheric oscillations (3 min period).

An impulsively excited wave obeying the Klein–Gordon equation leads to a wavefront travelling with the speed $c_t$ and trailing behind it a wake which oscillates at the cutoff frequency $\Omega$. Thus, we can expect a disturbance to propagate slightly sub-sonically, at speed $c_t$, and trailing a wake with an oscillation period of some 70 min. This is so long that it is difficult to observe, though the propagation speed of $c_t$ remains detectable.

Consider a driven oscillation prescribed at the footpoint ($z=0$) of a loop

$$Q = Q_0 \cos(\omega_0 t),$$

(4.7)

where $Q_0$ is the amplitude and $\omega_0$ the frequency of the driver. The solution of the Klein–Gordon equation (subject to an outgoing wave condition) may be formulated (Rae & Roberts 1982); we are only interested in the case $\omega_0 > \Omega$ here, since in the corona $\Omega$ is small. For large times $t$, we have a solution of the form

$$Q \sim Q_0 \cos \omega_0 \left[ t - \frac{z}{c_g} \right], \quad (z \ll c_t t, \, \omega_0 > \Omega),$$

(4.8)

where $c_g$ is the group velocity of the wave, determined by the frequency $\omega_0$ of the driver

$$c_g = \left( \frac{\omega_0^2}{\omega_0^2 - \Omega^2} \right)^{1/2} c_t.$$ 

(4.9)

This describes a wave moving vertically with the group speed $c_g (> c_t)$, the base ($z=0$) oscillating according to (4.7). With $\omega_0 = 0.035 \, s^{-1}$ (corresponding to a 3 min period) and $\Omega = 1.5 \times 10^{-3}$ (corresponding to a 70 min period), we obtain a speed of propagation which is slightly above the tube speed $c_t$. This effect may be difficult to detect in the corona.

5. Conclusions

The solar atmosphere supports a large variety of oscillatory phenomena, covering a range of frequencies. Sunspots, plumes, prominences and coronal loops are all observed to support propagating or standing waves, with both slow and fast magnetoacoustic waves arising. We have shown here how to extract out the behaviour of the slow wave from the MHD equations, separate from the fast magnetoacoustic wave. The result is a Klein–Gordon equation. Direct application of this theory to SUMER and TRACE observations suggests the effects of stratification are generally small. However, the approach we have demonstrated here for a vertical magnetic field may in principle be generalized to non-uniform magnetic fields, nonlinearity, flows and dissipative effects. Such effects may well be important.

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