Current-voltage relationship in downward field-aligned current region

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1. Introduction

[2] In this paper, we study the current-voltage relation on auroral field lines carrying a downward current. We find that the \( B/n \) peak plays a crucial role in determining Ohm’s Law, in particular the values of the current density (\( j_p \)) and number density (\( n_p \)) there. Other important quantities are the magnetospheric electron temperature (\( T_e \)), and the electron charge (\( e \)) and mass (\( m \)). We find that when the quantity \( \Psi = (\frac{e^2m}{2kTe^2})^{1/3} < 1.2 \), the total potential drop along the field line is approximately

\[
-\phi_{\text{av}} \approx \frac{3}{2} \left( \frac{j_p m^{1/2}kT}{n_pe^{5/2}} \right) + \frac{1}{2} \left( \frac{j_p m^{3/2}kT}{n_e^{3/2}} \right)^{1/2} + \frac{j_p m}{6n_pe^3} \tag{1}
\]

and when \( \Psi > 1.2 \)

\[
-\phi_{\text{av}} \approx \frac{kT}{e} \ln \left( \frac{j_p^2}{n_pe^2} \right) + \frac{j_p m}{2n_pe^3} + \frac{kT}{e} \tag{2}
\]

Exact expressions are also given. Our relations are compared with FAST observations, and show good agreement.

[3] Field-aligned currents form an integral part of the global magnetospheric current system, and are the means by which momentum is transferred between different plasma environments: the hot, tenuous magnetosphere and the cold, dense ionosphere. Field-aligned currents (FACs) are known to couple other space environments, including, for example, Io and Jupiter; however, those flowing along the Earth’s magnetic field lines are the easiest to observe at close range, giving vital clues as to the nature of these currents and the particle acceleration associated with them.

Several satellites and rockets, including FREJA [Marklund et al., 1994] and Viking, observed occasional upward accelerated field-aligned electron beams, but it was only with the advent of the Fast Auroral SnapshoT (FAST) satellite in 1996 that it was discovered that upward electron beams occur with much the same frequency as their downward counterparts [Carlson et al., 1998]. These upward beams are associated with diverging electric field structures, often located at the edge of the larger inverted V regions which support upward FACs; the improved time resolution and continuous observance of all pitch angles on FAST have been able to detect these small-scale dynamic downward current regions.

[4] Field-aligned currents are mainly carried by electrons. Observations of the downward current region by Andersson et al. [2002] and Ergun et al. [2003] have revealed complex characteristics: the potential increase occurs along a narrowly confined region of \( \sim 10 \) Debye lengths; the resulting unstable electron beam is seen along another similarly small region; and finally, the beam is stabilized by strong wave turbulence and electron phase space holes. Ion conics are also observed earthward of the potential structure, trapped between this and their mirror point. However, recent analysis has also shown examples of correlated increases in electron energy and potential, the latter of which is calculated from \( \int \mathbf{E} \cdot ds \) [e.g., Ergun et al., 1998]. These observations suggest that the potential structures are stable at least on the electron acceleration timescale, thus lending credence to the existence of quasi-static parallel potential structures. Our model gives a simple overview of this region based on this observational evidence. Detailed studies of the downward current region have shown that upward beams occur with greatest frequency in the winter hemisphere [Elphic et al., 2000], suggesting that the ion scale height and number density play a key role in determining the altitude and magnitude of the potential increase required [Cattell et al., 2004].
Several approaches have been employed to model the upward current region. Knight [1973] derived a linear current-voltage relation for the upward current region, where parallel potential drops are necessary to enable downflowing electrons to overcome magnetic mirror forces to carry the current. More recent models include Rönnmark [2002], Vedin and Rönnmark [2004] and Wright and Hood [2003]. Relatively little attention, however, has focused on the electron dynamics of the downward current region. It was thought that no significant potential would be required here, as the ionosphere is a plentiful source of electrons; however, the falling ion number density restricts the electron beam’s number density as it flows upward in such a way that it must be accelerated at a particular critical altitude in order to carry the required current. Temerin and Carlson [1998] use an electron fluid model with fixed ion density, invoking quasi-neutrality to calculate the required parallel potential drop. They obtain parallel potential drops of several kV for current densities of a few µAm⁻². Jasperse [1998] presents a Vlasov model including ion heating and wave effects, which explains the production of upward field-aligned electron beams and ion conics.

In paper 1, we analyze the downward current model presented in a separate paper in this special section [Cran-McGreehin and Wright, 2005, hereinafter referred to as paper 1], which builds on the Temerin and Carlson [1998] model by using electron distribution functions and modeling the entire F region. Mathematical analysis of this downward current model leads to a single current-voltage relation which can be simplified to give two nonlinear approximations valid under different regimes of the downward current region.

2. Model

In this paper, we present results obtained from the model in paper 1. This is a one-dimensional Vlasov model of an upward accelerated electron beam along an auroral field line, extending from the base of the F region, \( \ell_m \), taken to be at a radial distance of 1 \( R_E \), to a distant point in the magnetosphere, \( \ell_0 \). We model the magnetic field as being locally dipolar in the acceleration region, giving a magnetic field strength of

\[
B = B_0 \sqrt{1 + 3 \sin^2 \theta \cos^6 \theta} \tag{3}
\]

where \( B_0 \) is a constant, \( \theta \) is the latitude and

\[
r = L R_E \cos^2 \theta \tag{4}
\]

where we take \( L = 10 \) to give a typical auroral field line. The arc length element along \( B \), \( d\ell \), is given by

\[
d\ell = d\theta \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \tag{5}
\]

Using equations (4) and (5), the length along the field line, \( \ell \), which increases as you enter the ionosphere, is found to be

\[
\ell = \int_0^\theta L R_E \cos 0 \sqrt{1 + 3 \sin^2 \theta} \tag{6}
\]

which can be solved to give

\[
\ell = \frac{L R_E}{2 \sqrt{3}} \left( \frac{1}{2} \sinh^2 \left( \frac{1}{3} \sin^{-1} \left( \frac{1}{\sqrt{3}} \sin \theta \right) \right) + \sin^{-1} \left( \frac{1}{\sqrt{3}} \sin \theta \right) \right) \tag{7}
\]

The ion number density is considered to be fixed in time, and has an exponentially decaying profile

\[
n = n_0 + (n_m - n_0) \exp \left( -\frac{r - R_E}{h} \right) \tag{8}
\]

where the ion number densities at \( \ell_m \) and \( \ell_0 \) are \( n_m \) and \( n_0 \), respectively, and the scale height \( h \) ranges from 50 to a few hundred km. The scale height could simply represent the gravitational stratification of the ions, or could be increased in the case of strong ion conics to represent redistribution of ions by the mirror force [e.g., Jasperse, 1998]. A top-hat ionospheric electron distribution is defined at \( \ell_m \), and a Maxwellian magnetospheric electron population at \( \ell_0 \). Most of the ionospheric electrons are trapped by a small ambipolar electric field, but the most energetic ones escape and are accelerated into the magnetosphere to form the beam. We define \( \ell_c \) to be the point earthward of which all ionospheric electrons except those forming the beam are trapped. This location is important in the mathematical analysis and is where different solutions must be matched. For the purposes of this paper we define the “ionosphere” and “magnetosphere” in terms of this location: the ionosphere extends from \( \ell_m \), the base of the F region, to \( \ell_c \), and the magnetosphere is between \( \ell_c \) and \( \ell_0 \). The location of \( \ell_c \), along with other significant locations in the model, are shown schematically in Figure 1. The lower boundary of the model by Temerin and Carlson [1998] is at \( \ell_c \), where they input appropriate boundary conditions obtained from data. Our lower boundary is the base of the F region, \( \ell_m \), and the location of \( \ell_c \), where the electron beam emerges into the magnetosphere, is determined self-consistently from the model parameters (see paper 1). In paper 1, we reproduced Temerin and Carlson’s [1998] result for a specific example (see their Figure 1) by mapping their boundary conditions at \( \ell_c \) to \( \ell_m \). This reproduced their solution exactly, and gives confidence that our extended calculation is correct.

A full derivation of the equations is given in paper 1, to which the reader is referred for full details. Here, we outline the main features. The system is solved by using Liouville’s theorem and the current continuity condition, \( \nabla \cdot j = 0 \), which reduces to

\[
\frac{j(\ell)}{B(\ell)} = \frac{j_m}{B_m} \tag{9}
\]

where \( j_m \) and \( B_m \) are the values of the current density and magnetic field strength at \( \ell_m \). Expressions for the electron number density in the ionosphere and magnetosphere are obtained, and quasineutrality is then invoked to yield dimensionless equations. These contain three dimensionless parameters: a normalized current density, \( \alpha \), which is positive for downward currents, given by

\[
\alpha = \frac{j_m}{n_m e d_m} \tag{10}
\]
Diagram showing the three electron populations: mirroring magnetospheric electrons, trapped ionospheric electrons, and the beam. This diagram is not to scale: The magnetospheric electron population is much more energetic (∼1 keV) than that in the ionosphere (∼1 eV). Key locations derived from the model are \( \ell_m \), the base of the F region; \( \ell_e \), the ionospheric trapping point; \( \ell_p \), the location of the \( B/n \) peak; \( \ell_q \), the stationary point described in section 3; and \( \ell_o \), a distant reference point in the magnetosphere.

where \( a_m \) is the ionospheric electron distribution thermal velocity width; \( \eta \), the ratio of the ionospheric and magnetospheric temperatures, given by

\[
\eta = \frac{ma_m^2}{2kT}
\]  

where \( kT \) is the thermal energy of the magnetospheric electron population; and a normalized electric potential difference, \( \Delta \Phi(\ell) \), relative to that at \( \ell_m \), given by

\[
\Delta \Phi = \frac{2e}{ma_m}(\phi - \phi_m)
\]  

where \( \phi_m \) is the potential at \( \ell_m \), and \( \phi(\ell_o) = 0 \). The ionospheric and magnetospheric equations are shown below:

\[
\frac{n(\ell)}{B(\ell)} \bigg/ \frac{n_0}{B_0} = A \left( \sqrt{(1+2\alpha)^2 + \Delta \Phi} + \sqrt{1 + \Delta \Phi} \right)
\]  

\[
\frac{n(\ell)}{B(\ell)} \bigg/ \frac{n_0}{B_0} = A \left( \sqrt{(1+2\alpha)^2 + \Delta \Phi} - \sqrt{1 + \Delta \Phi} \right) + \frac{B_0}{B(\ell)} \left( 1 - AC \right) \exp(\eta(\Delta \Phi + \Phi_m))
\]

where

\[
A = \left( \frac{n_m}{n_0} \right) \frac{B_0}{B_m} \frac{1}{2(1+\alpha)}.
\]

and

\[
C = \sqrt{(1+2\alpha)^2 - \Phi_m} - \sqrt{1 - \Phi_m}
\]

[11] In the magnetospheric equation (14), the first term on the right hand side corresponds to the number density contribution from the electron beam, while the second term gives the contribution from the mirroring Maxwellian magnetospheric electron population. \( \Phi_m = 2e\phi_m/(ma_m^2) \) in equations (14) and (16) corresponds to the total change in potential along the field line, since using equation (12), \( \Delta \Phi(\ell_m) = 0 \) and \( \Delta \Phi(\ell_o) = -\Phi_m \). For downward currents, \( \Phi_m \) is negative, so the total potential increase along the field line is given by \( -\Phi_m \). Following Temerin and Carlson [1998] we have neglected the magnetospheric Maxwellian electron number density in the ionosphere (equation (13)) compared to the ionospheric electron number density. Paper 1 confirmed that this is a good approximation. If an additional term representing magnetospheric electrons was added to the RHS of equation (13), this would typically increase the relative electron number density by \( 10^{-4} - 10^{-5} \), and as suggested by Temerin and Carlson [1998] would have no significant effect on our solution. Indeed, we used the methods of paper 1 to find the exact numerical solution to (13) and (14) for standard parameters, with and without the inclusion of magnetospheric electrons in (13). This changed the altitude of \( \ell_e \) by 0.003% and \( \Phi_m \) by 0.06%, confirming the suitability of equation (13).

[12] In paper 1, we solve equations (13) and (14) numerically at each point along the field line to give a potential curve, in a similar fashion to Temerin and Carlson [1998]. In the ionosphere, we find that \( \Delta \Phi \) decreases from 0 to \( -1 \) between \( \ell_m \) and \( \ell_e \), this forms the ambipolar electric field which traps most of the ionospheric electron population close to the Earth. The most energetic electrons, although decelerated by this ambipolar potential, manage to penetrate into the magnetosphere to form the upcoming current-carrying beam. Beyond \( \ell_e \), the potential increases monotonically, and the parallel electric field maximizes at \( \ell_p \), between \( \ell_e \) and the \( B/n \) peak (see Figure 1). This kicksstart the electron beam acceleration, most of which occurs in a small acceleration region of width ∼1000 km. Qualitatively, results from paper 1 show that as the current density increases, so does \( |\Phi_m| \), the potential increase required to accelerate the beam; \( |\Phi_m| \) also increases as the difference in electron ionospheric and magnetospheric temperatures increases, which corresponds to a decrease in \( \eta \).

3. Stationary Point Analysis

3.1. Magnetospheric Roots

[13] We know the value of \( n/B \) along the field line from equations (3) and (8), so in the ionosphere, we can solve equation (13) at each point between \( \ell_m \) and \( \ell_e \) to find the potential, \( \Delta \Phi(\ell) \). Solving equation (14) to find the potential variation in the magnetosphere is not so straightforward, as the equation contains a free parameter \( \Phi_m \), where \( \Delta \Phi(\ell_o) = -\Phi_m \). Varying this parameter produces a family of curves of the roots of \( \Delta \Phi \) in equation (14); Temerin and Carlson [1998] also noted the multivalued nature of their solution.

[14] In general, the solutions to this equation are described by two branches, or curves, for a given \( \Phi_m \), as shown in Figure 2. For one particular critical value of \( \Phi_m \), these branches touch at a stationary point. If we take \( |\Phi_m| < \Phi_m \), then there are two branches, one to the right of
The stationary point at \( \ell_q \) is a standard two-variable saddle point. At a stationary point of a general function \( g(x, y) \), \( \frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0 \) [see Salas et al., 2003, chapter 15]. Equation (18) tells us that \( G(B, \Delta \Phi, \Phi_m) = 0 \), where \( n \) is a known function of \( B \) and all other parameters are given, but \( \Phi_m \) is not (hence it is retained as a variable). Setting \( G = 0 \) implicitly defines \( \Phi_m \) given \( B \) and \( \Delta \Phi \), which we can express as \( \Phi_m = g(B, \Delta \Phi) \). We see how \( B \) and \( \Delta \Phi \) play the roles of \( x \) and \( y \) in a textbook two-variable stationary point analysis. Indeed, the contours of \( \Phi_m \) in Figure 2 were calculated from just this expression (except it uses \( s \) rather than \( B \) as the field-aligned coordinate). The requirements \( \frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0 \) become \( \frac{\partial g}{\partial B} \Delta \Phi_m = \frac{\partial g}{\partial B} (\Delta \Phi)_{\Delta \Phi = 0} = 0 \) and \( \frac{\partial g}{\partial \Delta \Phi} B_m = 0 \) at \( \ell_q \), giving us two more equations for the stationary point:

\[
0 = -\frac{1}{2} \left( \frac{n_m}{n_0} \right) \left( \frac{B_q}{B_m} \right) \frac{\alpha}{(1 + \Delta \Phi_q)^2} + n(1 - AC) \exp(n(\Delta \Phi_q + \Phi_m))
\]

valid when \( 1 + \Delta \Phi \gg 4 \alpha (\alpha + 1) \). Since \( \alpha \ll 1 \), typically \( \sim 10^{-5} - 10^{-3} \), and our stationary point \( \ell_q \) lies in the middle of the acceleration region where \( \Delta \Phi \) is large, this is an excellent approximation. Indeed, for typical parameters, \( 4\alpha (\alpha + 1) (1 + \Delta \Phi) \sim 10^{-7} - 10^{-6} \). From this approximation, we obtain

\[
\frac{n(\ell)}{B(\ell)} / n_0 = \left( \frac{n_m}{n_0} \right) \left( \frac{B_q}{B_m} \right) \frac{\alpha}{\sqrt{1 + \Delta \Phi}}
\]

\[
+ \frac{B_0}{B(\ell)} (1 - AC) \exp(n(\Delta \Phi + \Phi_m))
\]

3.2. Stationary Point Equations

In paper 1 we showed numerically that the \( B/n \) peak determined the solution; altering the ion number density on either side of the peak resulted in no difference to \( \Phi_m \). We now understand why this is. The stationary point outlined above can be described mathematically. To do this, we make a small simplification to equation (14) to allow solution of the resulting equations, by using the approximation

\[
\sqrt{(1 + 2\alpha)^2 + \Delta \Phi} - \sqrt{1 + \Delta \Phi} \approx \frac{2\alpha(1 + \alpha)}{\sqrt{1 + \Delta \Phi}}
\]

Figure 2. Plot showing the form of the two roots of magnetospheric equation (14) for different values of the parameter \( \Phi_m \), where \( \alpha = 5 \times 10^{-5}, \eta = 10^{-3} \) and \( s = \ell - \ell_m \). If \( |\Phi_m| \) is larger than the critical value, then upper and lower branches of \( \Delta \Phi \) exist all the way along the field line (above and below the \( x \) point), but do not meet at any point, and neither curve satisfies our boundary conditions. If \( |\Phi_m| \) is smaller than the critical value, then there are again two branches of \( \Delta \Phi \), one to the right and the other to the left of the \( x \) point. Thus there are points along the field line between these two curves for which no root of \( \Delta \Phi \) exists. All of these curves are unphysical, but there is a unique critical value of \( \Phi_m \) for which the two curves of \( \Delta \Phi \) meet at a stationary \( x \) point. The solution which satisfies our boundary conditions is the monotonically increasing one, which switches from the lower to the upper branch as it passes through the \( x \) point.

In principle, we could solve this system of equations, by introducing a fourth equation relating \( n_q \) and \( B_q \). However, this proves to be unnecessary, as we shall show in section 4 that it is possible to get good estimates of \( n_q \) and \( B_q \) by another route. We do not use equation (21) in our derivation of an analytical expression for \( \Phi_m \), but we have included it for completeness, since it would be necessary to use it if \( n_q \) and \( B_q \) were unknown.
Substituting for the exponential in equation (19) using equation (20) we obtain
\[ \frac{\eta_q}{\eta_0} = \frac{\eta_m}{\eta_0} B_{a_s} \frac{\alpha}{\sqrt{1 + \Delta \Phi_q}} + \frac{\alpha \eta_m B_{a_s}}{2 \eta_0} \frac{1 + \Delta \Phi_q^{3/2}}{2} \]  
(22)

where typically, \( \eta \approx 1 \). This can be rearranged to give the cubic
\[ X^3 - \alpha \eta_q X^2 + \frac{\alpha \eta_q}{2} = 0 \]  
(23)

where \( X = \sqrt{1 + \Delta \Phi_q} \) and \( \eta_q = (\eta_m \eta_q) (B_{a_s} / B_m) \).

3.3. Solving the Cubic

The cubic equation (23) can be solved by first transforming it:
\[ X^3 - \alpha \eta_q X^2 + \frac{\alpha \eta_q}{2} = \left( X - \frac{\alpha \eta_q}{3} \right)^3 - \frac{\alpha^2 \eta_q^2}{3} \left( X - \frac{\alpha \eta_q}{3} \right) \]  
- \left( \frac{\alpha \eta_q}{2} + \frac{\alpha \eta_q^3}{27} \right) \]  
(24)

Now we can use the fact that a cubic of the form \( y^3 + py + q = 0 \) has a real solution \( y = s - t \), where \( p = 3st \) and \( q = t^3 - s^3 \). In the case of equation (24), \( Y = X - (\alpha \eta_q / 3) \), and this gives
\[ \sqrt{1 + \Delta \Phi_q} = \sqrt[3]{\frac{\alpha \eta_q}{2 \eta_0}} - \frac{\alpha \eta_q}{3} \sqrt{1 + \frac{8 \eta_0^2 \epsilon_q^2}{27}} \]  
(25)

We are now in a position to derive an expression for \( \Phi_m(\alpha, \eta, n_q, B_{a_s}) \) by rearranging equation (20) to give
\[ \Phi_m = \frac{1}{\eta} \ln \left[ \frac{\alpha n_q B_{a_s}}{2 \eta_0} \left( 1 - AC \right)^{-1} \left( 1 + \Delta \Phi_q^{3/2} \right)^{-1} \right] - \Delta \Phi_q \]  
(26)

We can now substitute equation (25) into equation (26) to obtain
\[ \Phi_m = \frac{1}{\eta} \ln \left[ \frac{2 n_q}{n_0 (1 - AC)} \right] - \frac{3}{\eta} \ln [y(\Psi)] \]  
- \left( \frac{\alpha \eta_q}{4 \eta_0} \right)^{2/3} [y(\Psi)]^2 + 1 \]  
(27)

where
\[ y(\Psi) = \sqrt{1 + 4 \Psi^3 / 27} - \sqrt{1 + 8 \Psi^3 / 27} \]  
+ \sqrt{4 \Psi^3 / 27 + \sqrt{1 + 8 \Psi^3 / 27} + 4/3 \Psi} \]  
(28)

and
\[ \Psi = \alpha^{2/3} \epsilon_q^{2/3} \eta_1^{1/3} \]  
(29)

The constant \( C \), contained in our expression for \( \Phi_m \) in equation (27) and defined in equation (16), depends on \( \Phi_m \). However, we can split up the first term on the RHS of equation (27) as follows:
\[ - \frac{1}{\eta} \ln \left( \frac{2 n_q}{n_0 (1 - AC)} \right) = \frac{1}{\eta} \ln \left( \frac{2 n_q}{n_0} \right) - \frac{1}{\eta} \ln (1 - AC) \]  
(30)

where typically, \( \eta^{-1} \ln (1 - AC) / \Phi_m \sim 10^{-4} \). Thus we can safely neglect this term to obtain a more useful expression for \( \Phi_m \):
\[ \Phi_m = \frac{1}{\eta} \ln \left( \frac{2 n_q}{n_0} \right) - \frac{3}{\eta} \ln [y(\Psi)] \]  
- \left( \frac{\alpha \eta_q}{4 \eta_0} \right)^{2/3} [y(\Psi)]^2 + 1 \]  
(31)

When \( \Phi_m \) is evaluated using equation (31) and the exact value of \( \epsilon_q \), taken from a numerical solution, we find exact agreement with the numerically derived value of \( \Phi_m \), as expected. The results of such exact calculations for a range of values of \( \alpha \) and \( \eta_0 \) are displayed in Figure 3. The expression in equation (31) is helpful because although \( \Phi_m \) is a function of two variables, \( \alpha \) and \( \eta_0 \), the expression contains \( y(\Psi) \), a function of one variable which is easier to expand in a Taylor series to simplify the expression.

4. Taylor Series Approximations

4.1. Approximation of \( \epsilon_q \)

It is very hard to pinpoint the exact location of \( \epsilon_q \) analytically by solving the stationary point equations. However, the numerical solutions of paper 1 can be used to show that \( \epsilon_q \) always lies very close to the \( B/n \) peak. The results of these numerical solutions are shown in Figure 4 for varying ion scale heights \( h \), current densities and electron temperatures. The ion scale height varies with ion temperature, which can range from 10^3 K to a few thousand K: this gives ion scale heights from 50 to a few hundred km. The values of the two small parameters, \( \alpha \) and \( \eta_0 \), do have a small effect on the location of \( \epsilon_q \) as \( \alpha \) (the normalized current density) increases, \( \epsilon_q \) moves closer to the \( B/n \) peak; and as the ratio of ionospheric and magnetospheric electron temperatures, \( \eta_0 \), increases (implying a lower magnetospheric temperature), \( \epsilon_q \) again moves closer to the \( B/n \) peak. However, in all applicable cases, the difference in height is no more than 10%.

As a result of this, we can make the approximations \( \eta_q \approx \eta_0 \) and \( B_{a_s} \approx B_p \) in the analytical solution, and obtain a very accurate approximation to \( \Phi_m \). Thus we can approximate the exact solution by replacing \( \epsilon_q \) in equation (29) with
\[ c_p = \left( \frac{n_q}{n_p} \right) \left( \frac{B_{a_s}}{B_m} \right) \]  
(32)

Figure 4 shows a linear relationship between the \( B/n \) location and ion scale height, \( h \). This work illustrates that, not surprisingly, the ion scale height plays the major role in determining the position of the \( B/n \) peak, and hence of \( \epsilon_q \). This agrees with work on the prevalence of beams in the
4.2. Taylor Series

Equation (31) constitutes an analytical solution of $\Phi_m$, the total normalized potential increase along the field line, in terms of $\alpha$, $\eta$, $c_q$ and $n_q$. However, it is a cumbersome expression, and it would be very helpful to obtain more user-friendly approximations for use in future analytical work. This can be achieved by performing a Taylor series expansion on $\gamma(\Psi)$, given in equation (28), and then substituting this expansion into equation (31) to obtain an approximate relation.

4.2.1. $\Psi \ll 1$

First, we assume that $\Psi = \alpha^{2/3} c_q^{2/3} n_q^{1/3}$ is small. This corresponds to cases where $\alpha$ and $\eta$ are both small, implying small current densities and moderate to high...
magnetospheric temperatures. Performing a Taylor expansion of equation (28) around $\Psi = 0$ gives us

$$y(\Psi) = 2^{1/3} + \frac{4^{1/3}}{3} \Psi + \frac{2}{9} \Psi^2 + \frac{4}{81} \Psi^{4/3} + O(\Psi^4)$$

(33)

[30] Substituting this back into equation (31) and using equation (29) with $c_q \approx c_p$, we obtain the following expression for $\Phi_m$

$$\Phi_m(\alpha, \eta) \approx \frac{1}{\eta} \ln \left( \frac{n_p}{n_0} \right) - 3 \left( \frac{\alpha c_p}{2 \eta} \right)^2 - \frac{1}{2} \ln \left( \frac{\alpha^{4/3} c_p^{4/3}}{\eta^{4/3}} \right) - \frac{\alpha^2 c_p^2}{3} + O\left( \alpha^{1/3} c_p^{4/3} \eta^{4/3} \right)$$

(34)

[31] It is possible to neglect the first term on the right hand side, since $n_p \approx n_0$. Thus, noting that $\alpha$ is positive for downward currents, the expansion can be simplified to

$$-\Phi_m \approx 3 \left( \frac{\alpha c_p}{2 \eta} \right)^2 + \frac{1}{2} \ln \left( \frac{\alpha^{4/3} c_p^{4/3}}{\eta^{4/3}} \right) + \alpha^2 c_p^2$$

(35)

4.2.2. $\Psi \gg 1$

[32] The other possibility is that $\Psi = \alpha^{2/3} c_p^{2/3} \eta^{4/3}$ is large. This implies large values of $\alpha$ and $\eta$, i.e., large current densities and low magnetospheric temperatures. Performing a Taylor expansion of equation (28) about $\Psi = 0$ is equivalent to expanding $y(1/\Psi)$ about $1/\Psi = 0$. So, letting $a = 1/\Psi$, we can manipulate equation (28) to obtain

$$Y(a) = \left( \frac{1}{a} \right) \left( \frac{4}{27} + a^3 - \sqrt{\frac{8}{27} a^{1/2}} \sqrt{1 + \frac{27}{8} a^3} \right) + \frac{1}{2} \left( \frac{4}{27} + a^3 + \sqrt{\frac{8}{27} a^{1/2}} \sqrt{1 + \frac{27}{8} a^3 + \frac{47}{3}} \right)$$

(36)

[33] Expanding the bracket around $a = 0$ yields

$$aY(a) = 4^{1/3} + a^3 + O(a^4)$$

(37)

which can be translated to

$$y(\Psi) = 4^{1/3} \Psi + \frac{1}{2} \eta^{-1/3} \Psi^3 + O(\Psi^4)$$

(38)

[34] As above, we can substitute this into equation (31) to obtain

$$\Phi_m(\alpha, \eta) \approx \frac{1}{\eta} \ln \left( \frac{2 n_p}{n_0} \right) - \frac{1}{\eta} \ln \left( 4 \alpha^2 c_p^2 \eta \right) - \alpha^2 c_p^2$$

$$- \frac{1}{\eta} + O\left( \alpha^{-2/3} c_p^{-2/3} \eta^{-4/3} \right)$$

(39)

[35] For the same reasons as in the first case, we can simplify the first term to $\eta^{-1} \ln(2)$ to obtain the following approximation

$$-\Phi_m \approx \eta \left( 2 \alpha^2 c_p^2 \eta + \alpha^2 c_p^2 + \frac{1}{\eta} \right)$$

(40)

4.3. Accuracy of Approximations

[36] Equations (31), (28) and (29) give the analytical form of $\Phi_m$ if $c_q$ is known. However, finding this parameter is awkward and involves numerical work, since the location of $\ell_c$ changes with $\alpha$ and $\eta$ for a given equilibrium model. If we let $c_q \approx c_p$, which is easier to calculate (a range of values are given in Table 1), we obtain approximate relations for $\Phi_m$ which are easy to use.

[37] The relative accuracy of the two approximations is shown in Figure 5 for two different values of $\eta$. In each case, $\Psi < 1$ for small current densities and the first approximation in equation (35) is accurate. Many relevant scenarios in the downward current region with low to moderate current densities and average ionospheric and magnetospheric electron temperatures satisfy $\Psi \ll 1$, so this approximation is valid. Even as $\alpha$ increases and $\Psi$ approaches 1 and exceeds it, this approximation remains very accurate. Then, at some value of $\Psi \approx 1$, the first approximation loses accuracy as we enter a different regime where the second approximation in equation (40) should be adopted. In all cases for the standard parameters we have chosen, the appropriate approximation is valid to within 6.4% of the exact potential increase.

4.4. An Example

[38] In order to estimate the potential increase for a given event in the downward current region, the following steps should be carried out.

[39] Step 1 is to choose the parameters for the equilibrium model: the magnetospheric thermal electron energy ($kT$), the ionospheric electron thermal energy ($m_e a_m^2/2$), the ion number density at the base of the F region ($n_m$) and at $\ell_0$ in the magnetosphere ($n_0$), the ion scale height ($h$), and the current density at the base of the F region ($j_m$). From these, calculate

### Table 1. Values of the Constant $c_p$ for Different Ion Number Densities ($n_n/n_0$) and Scale Heights ($h$)

<table>
<thead>
<tr>
<th>$n_n/n_0$</th>
<th>$h$ (km)</th>
<th>$c_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>10</td>
<td>0.01</td>
</tr>
<tr>
<td>0.2</td>
<td>20</td>
<td>0.02</td>
</tr>
<tr>
<td>0.3</td>
<td>30</td>
<td>0.03</td>
</tr>
<tr>
<td>0.4</td>
<td>40</td>
<td>0.04</td>
</tr>
<tr>
<td>0.5</td>
<td>50</td>
<td>0.05</td>
</tr>
</tbody>
</table>

*The first row shows values of $n_n/n_0$, and the first column gives different values of $h$. 
the dimensionless parameters $\alpha$ and $\eta$, from equations (10) and (11), respectively.

[40] Step 2 is to determine the parameter $c_p$ given in equation (32). This can be done by using Table 1. Also determine the value of $\Psi$ in equation (29), replacing $c_q$ with $c_p$.

[41] Step 3 contains two options. The first is to use equation (28) to determine $\Psi$ and substitute into equation (31), using $n_q \approx n_0$ and $c_q \approx c_p$ to obtain $\Phi_m$. The second is to use the approximation in equation (35) if $\Psi > 1$, or the one in equation (40) if $\Psi < 1$. Note that $\Phi_m$ is negative for downward currents, and $-\Phi_m$ gives the total potential increase along the field line.

[42] We now use this approach to approximate the potential in the FAST data given by Carlson et al. [1998, Figure 2]. This is taken at an altitude of around 3965 km, which we denote by $\ell_{\text{FAST}}$. The current density at this altitude, $j_{\text{FAST}}$, varies from 1 to 2.5 $\mu$A/m$^2$, so we consider both of these cases. We use the current continuity condition in equation (9) to obtain corresponding values for $j_m$, which are 4.29 (case 1) and 10.7 (case 2) $\mu$A/m$^2$. These are high current densities, so this is a strong downward current event. We take typical ionospheric and magnetospheric electron temperatures of 1 eV and 1 keV, giving $\eta = 10^{-3}$; $n_m$ is taken to be $10^{11}$ m$^{-3}$, slightly lower than used previously to account for the fact that this event occurred at night, giving $n_m/n_0 = 10^5$; finally, the ion scale height is taken to be 150 km, slightly larger than before to account for transverse ion heating effects which modify the distribution of ions. The results are shown in Figure 6; the total potential increase is 1730 V for

Figure 5. Graphs showing the regions of validity of the two Taylor expansions. The current densities run from 1 to 10 $\mu$A/m$^2$; for an ionospheric electron temperature of 1 eV, $\eta$ values of $10^{-3}$ and $2.5 \times 10^{-3}$ correspond to magnetospheric electron temperatures of 1 keV and 400 eV, respectively. (top) The solid line represents the actual values of $\Phi_m$ found by solving equation (14) numerically, or using the numerically determined values of $n_q$ and $B_q$ in equation (31). The dashed line shows the approximation given in equation (35), which is most accurate for small $\alpha$ and $\eta$, while the dot-dashed line shows the approximation given in equation (40), most accurate for larger values of $\alpha$ and $\eta$. (bottom) The corresponding values of $\Psi$ are plotted, defined in equation (29), which is the variable in which we derive the Taylor series. The vertical solid line indicates the point at which $\Psi = 1$ in each case. The first approximation is more accurate for $\Psi \leq 1.2$. When $\Psi \geq 1.2$, the second approximation should be adopted.
We obtain:

$$
\Phi_m = \frac{m}{2e} \left( \frac{\mu_B}{\hbar} \right)^{2/3} \psi_{th}^{1/3} + \frac{u_p^{4/3} \psi_{th}^{2/3}}{2v_{th}^{3}} + \frac{u_p^2}{3}
$$

(46)

5. Potential Drop in Terms of Current Densities

5.1. Potential Drop in Terms of Speeds

We can derive expressions for $\psi_m$, the actual potential increase along the field line, in terms of two characteristic speeds: the mean electron drift speed at the $B/n$ peak, $u_p$, defined as

$$
u_p = \frac{j_p}{n_\psi e}
$$

(43)

where $j_p = j_n B_p / B_m$, and the magnetospheric thermal velocity, $v_{th}$, given by

$$
\nu_{th}^2 = \frac{2kT}{m}
$$

(44)

If we substitute equations (43) and (44) into equation (29), we find that

$$
\psi^3 = \left( \frac{\nu_p}{\nu_{th}} \right)^2
$$

(45)

Thus the small-$\Psi$ expression for $\Phi_m$ in equation (35) corresponds to cases where $u_p \ll \nu_{th}$, i.e., for relatively small current densities and moderate to high magnetospheric electron temperatures. We substitute equations (43) and (44) into (41) to obtain an expression for $\psi_m$ in this regime:

$$
-\psi_m = \frac{m}{2e} \left( \frac{\mu_B}{\hbar} \right)^{2/3} \psi_{th}^{1/3} + \frac{u_p^{4/3} \psi_{th}^{2/3}}{2v_{th}^{3}} + \frac{u_p^2}{3}
$$

(46)

In the alternative regime, where $\Psi > 1$ or $u_p \gg \nu_{th}$, implying higher current densities and lower magnetospheric electron temperatures, we substitute equations (43) and (44) into (42) to obtain:

$$
-\psi_m = \frac{m}{2e} \left( \frac{\mu_B}{\hbar} \right)^{2/3} \psi_{th}^{1/3} + \frac{u_p^{4/3} \psi_{th}^{2/3}}{2v_{th}^{3}} + \frac{u_p^2}{3}
$$

(47)

5.2. Potential Drop in Terms of Current Densities

Alternatively, we can derive $\Phi_m$ in terms of two characteristic current densities: the beam current density at the $B/n$ peak, written using equation (9) as

$$
j_p = j_n B_p / B_m
$$

(48)
and the field-aligned thermal current density, \( j_{th} \), due to the downgoing component of the Maxwellian electron distribution (i.e., 0 < \( v_{\parallel} \) < \( \infty \), 0 < \( v_{\perp} \) < \( \infty \)) at \( \ell_0 \):

\[
j_{th} = -e \int_0^\infty v f_{th} dv = -e n_0 \sqrt{\frac{k T}{2 \pi m}}
\]

(49)

where

\[
f_{th} = n_0 \left( \frac{m}{2 \pi k T} \right)^{3/2} \exp \left( -\frac{m}{k T} \left( v_{\parallel}^2 + v_{\perp}^2 \right) \right)
\]

(50)

[54] Of course, we assume that these electrons mirror, so this population does not carry a net current. The parameter \( \Psi \) in equation (29) can now be written as

\[
\Psi^3 = \frac{1}{4 \pi} \frac{f_p n_h^2}{n_p^2}
\]

(51)

[55] So, the small-\( \Psi \) expression (41), valid when \( f_p \ll |j_{th}| \), can be written in terms of (48) and (49) to give

\[
-\phi_m = \frac{m}{2e^3} \left[ 3 \left( \frac{2 \pi}{n_0^2 n_p} \right)^{2/3} f_p^{2/3} |j_{th}|^{4/3}
\right.
\]

\[
+ \left. \left( \frac{2 \pi}{n_0^2 n_p^2} \right)^{1/3} f_p^{4/3} |j_{th}|^{1/3} - \frac{f_p^2}{3 n_p^2} \right]
\]

(52)

[56] Similarly, the large-\( \Psi \) expression in equation (42), valid when \( |j_{th}| \ll f_p \), can be written as

\[
-\phi_m = \frac{m}{e^3} \left( \frac{2 \pi}{n_0^2 n_p} \right) \ln \left( \frac{f_p n_h}{2 \pi n_0^2} \right) + \frac{f_p^2}{2 n_p^2} + \frac{2 \pi f_{\phi}^3}{n_0^2}
\]

(53)

[57] Further simplifications can be made to equations (52) and (53) by noting that \( n_p \approx n_0 \).

6. Discussion and Conclusions

[58] The analysis presented here reveals that the \( B/n \) peak is central to the solution of the downward current region. Mathematically, the equations contain a stationary point which lies slightly beyond the \( B/n \) peak. Examination of this stationary point yields a near-exact solution for \( \phi_m \), the total potential increase along the field line (given the model parameters, and \( n_h \) and \( B_p \)). Taylor series expansions of the exact solution can be carried out to give two simplified nonlinear current-voltage relations. Since the stationary point is so close to the \( B/n \) peak, approximations can be derived for which we use the number density and magnetic field strength at the \( B/n \) peak. One approximation is valid for lower current densities and moderate to high magnetospheric temperatures (~1 keV), while the other takes over for higher current densities and lower magnetospheric temperatures. Typically, the first expansion will be more useful for most downward current regions in the Earth’s magnetosphere, except for stronger events with particularly high current densities. Both of these expansions, when written in dimensional form, are independent of ionospheric equilibrium parameters, illustrating that the exact properties of the ionosphere are unimportant in this model.

[59] Observations suggest that acceleration in the downward current region can occur over a very small distance [Ergun et al., 2003] (i.e., a double layer). In this case, the change in potential along the field line is compacted, which can be achieved via a sharp decrease or change in ion number density at a certain altitude [Temerin and Carlson, 1998]. Just as Knight’s [1973] model of the upward current provides a good overview of the region as a whole, neglecting double layers, we describe the downward current region on a similarly large scale. Thus we find a smooth transition in potential, but it is quite possible that this overall change is confined to sharp increases in potential contained within several double layers [Andersson et al., 2002]. In any case, the overall change in potential is likely to be similar in both cases. In paper 1, we find that it is the properties of the small region surrounding the \( B/n \) peak (containing the stationary point \( \ell_q \)) which solely determine the total potential change \( \phi_m \). This has been corroborated by the analytical work presented here, which shows that we can determine \( \phi_m \) simply by solving the stationary point equations at \( \ell_q \). If we were to include ion motion in this model, it is possible that the ion distribution would steepen into a double layer, which would preserve the single \( B/n \) peak and effectively move \( \ell_q \) even closer to it: thus our analysis should still be appropriate.

[60] The location of the \( B/n \) peak is principally dependent upon the ion scale height and number density, \( n_m n_0 \). In Cattell et al.’s [2004] statistical survey of the occurrence of upward accelerated electron beams, many more beams were observed on field lines where the ionospheric foot point is in darkness than when it is illuminated. This makes sense in terms of the model, since at night time, the number density and scale height will be smaller because of lack of photoionization and cooler temperatures. These two factors cause the \( B/n \) peak, and hence the stationary point, to move earthward. With a smaller number density, the ionosphere will provide fewer charge carriers which, on encountering lower ion number densities due to the decreased scale height, will need to be accelerated at lower altitudes to meet the demands of quasineutrality and carry the required current.

[61] The current-voltage relations derived here show good agreement with observational data. Qualitatively, our model predicts potential increases of ~100 V to ~1 kV, which tally well with observations of downward current regions. Quantitatively, our model agrees very well with the FAST data from Carlson et al. [1998]. It will be desirable to make further comparisons with data to check consistency over a range of current densities.

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References


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